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BINDING STATES OF INDIVIDUAL NUCLEONS IN STRONGLY DEFORMED NUCLEI

BY

SVEN GÖSTA NILSSON

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i kommission hos Ejnar Munksgaard

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I. Introduction.

The nuclear shell model has had considerable success in recent years in accounting for various regularities in nuclear properties. In this model one considers the nucleons as moving independently in an averaged potential. For a particular nucleon this potential represents its interaction with all other nucleons in the nucleus. In particular, it has been possible by choosing an appropriate field, containing rather strong spin-orbit coupling, to obtain a succession of single particle states which reproduce the experimentally observed discontinuities associated with the so-called magic numbers.*

In the usual formulation of the shell model the potential is assumed to be isotropic, but it has been found that nuclei with proton and neutron numbers very different from those corresponding to closed shells have large deformations, as evidenced, e. g., by large quadrupole moments. The deformation of the nuclear field may have a great influence on the motion of the individual nucleons, and it is the aim of this paper to consider the binding states of nucleons in such a deformed potential.

The introduction of a non-spherical binding field implies that the nuclear shape and orientation must be considered dynamical variables. These variables are associated with the collective types of nuclear motion which accompany variations in the binding field. The interplay between these collective modes of motion and the individual-particle motion forms the basis of the unified nuclear model.**

* M. G. Mayer, Phys. Rev. **75**, 1969 (1949).
O. Haxel, J. H. D. Jensen, and H. E. Suess, Zs. f. Physik **128**, 295 (1950).
** A. Bohr, Dan. Mat. Fys. Medd. **26**, no. 14 (1952).
A. Bohr and B. Mottelson, Dan. Mat. Fys. Medd. **27**, no. 16 (1953).
A. Bohr, E. Munksgård, Copenhagen (1954).
In the following, these papers are referred to as AB, BM, and AB 1954, respectively.
Cf. also D. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).

The nuclear properties resulting from this interplay are found to depend essentially on the magnitude of deformation, which again depends on the nucleonic configuration. In the regions of major closed shells the equilibrium shape of the nucleus is spherical, and the individual-particle spectrum may be obtained by considering particle motion in a spherical field, as in the shell model. It is expected, however, that in these regions the nuclei have also additional modes of excitation of the collective vibrational type. The dependence of the particle motion on the nuclear shape implies for these nuclei a small interweaving of collective and particle motion, which may be described by a perturbation treatment. The further addition of particles leads to a larger nuclear deformation. The coupling between collective modes and individual-particle modes of motion may in such cases lead to a very complicated structure of nuclear states.

Still further from the closed shells, however, the situation again simplifies. The nucleus then acquires a large deformation with a resulting stability of orientation. It is then possible to separate approximately between intrinsic nucleonic motion relative to the deformed but fixed nuclear field, and the collective rotational and vibrational motion, which leaves unaffected the intrinsic structure.*

The separation of the different modes of motion is best evidenced empirically by the occurrence of rotational spectra, which are found to obey the simple theoretical expressions with remarkable accuracy.**

The separation of the nuclear motion into collective and intrinsic modes corresponds to the assumption of a wave function of the product type as solution to the nuclear wave equation

$$\Psi = \chi \cdot \varphi_{\text{vib}} \cdot \mathfrak{D}_{\text{rot}} \cdot$$

Here χ represents the intrinsic motion of the nucleons, which can be expressed in terms of the independent motion of the

* In AB, BM and AB 1954 this approximate solution of the equations of motion appropriate to strongly deformed nuclei is denoted "the strong coupling scheme".

** Cf., e. g., AB 1954 and A. Bohr and B. Mottelson, Chapter 17 of "Beta and Gamma Ray Spectroscopy", ed. by K. Siegbahn, North Holland Publishing Co. (1954).

individual particles in the deformed field, which is considered as stationary. The second factor, φ_{vib}, describes the vibrations of the nucleus around its equilibrium shape, while $\mathfrak{D}_{\text{rot}}$ represents the collective rotational motion of the system as a whole.

Most nuclei are expected to prefer shapes of cylindrical symmetry, and this is confirmed by the observed rotational spectra.* Therefore we here restrict ourselves to the consideration of particle states in fields of the spheroidal type.

In this case of axial symmetry, the intrinsic motion is characterized by the quantum numbers Ω_p, the component of angular momentum of each nucleon along the nuclear axis. The total Ω is given by $\Sigma\Omega_p$. Apart from accidental degeneracies, states are doubly degenerate (corresponding to $\pm\Omega_p$), and the total χ, in the following denoted χ_Ω, is therefore simply the antisymmetrized product of individual-particle wave functions χ_{Ω_p}. The presence of direct particle forces produces to first order a shift in the binding energies without, however, affecting the wave functions. The nucleonic coupling scheme will be essentially modified only if the particle forces are comparable with the coupling of individual particles to the nuclear axis.

The rotational motion is characterized by the quantum numbers I, M, and K, i. e. the total angular momentum, its projection on the space fixed axis (later denoted by z''), and its projection on the intrinsic nuclear axis (z'), respectively. (See Fig. 1.)

We shall not here be concerned with the corresponding vibrational quantum numbers, since we always assume that we are in the vibrational ground state.

Beside the rotational symmetry around the nuclear axis we also assume that the nucleus has reflection symmetry through a plane perpendicular to this axis. The wave function has then to be symmetrized (to possess a definite parity). The appropriately symmetrized wave function may be written in the form**

$$|\Omega,\ IMK\rangle = \sqrt{\frac{2I+1}{16\pi^2}}\,\varphi_{\text{vib}}\left\{\chi_\Omega\mathfrak{D}^I_{MK}(\theta_i) + (-)^{I-\Sigma j_p}\chi_{-\Omega}\mathfrak{D}^I_{M-K}(\theta_i)\right\}. \quad (1)$$

* There may be special configurations for which the cylindrically symmetric shape is not stable. The rotational spectra are then of a more complex character than in the symmetric case. Cf. B. SEGALL, Phys. Rev. **95**, 605 A (1954) and M. JEAN and L. WILETS (to be published).

** BM (II.15).

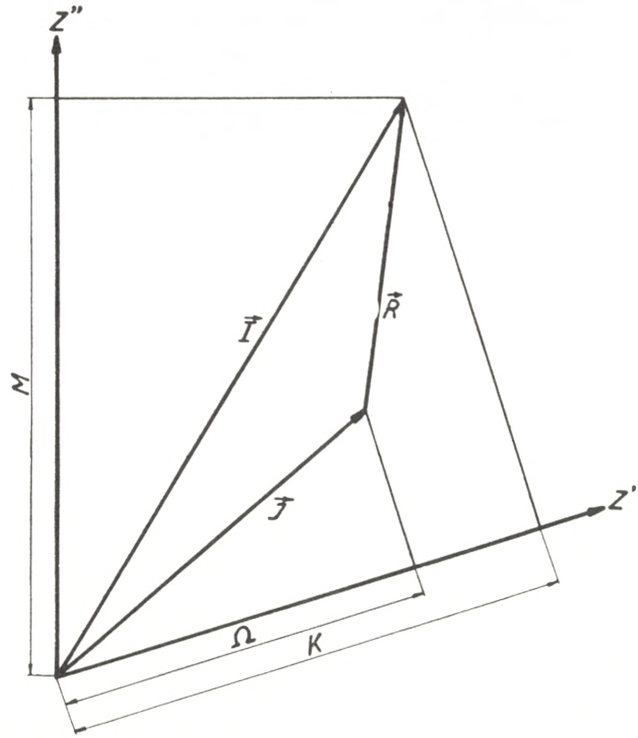

Fig. 1. *Angular momentum diagram.*

In the unified model the total angular momentum $\bar{I}$ is composed of two parts, one part $\bar{R}$ generated by the collective motion of the nucleus, the other part $\bar{J}$ representing the intrinsic motion of the nucleons.

In the coupling scheme appropriate for large deformations the nucleons move independently with respect to the deformed nuclear field. This motion is characterized by the constants of the motion Ω_p, the component of angular momentum of each nucleon along the nuclear axis. In such a structure the magnitude of the total $\bar{J}$ is not a constant of the motion, though its component on the nuclear symmetry axis is a good quantum number and is denoted Ω, where $\Omega = \sum_p \Omega_p$.

Finally the rotational state of the system is described in terms of the quantum numbers I, the total angular momentum, its z''-component M, and its z'-component K.

In the ground state, $\bar{R}$ is perpendicular to z' ($\Omega = K$), i. e. the collective rotation takes place around an axis perpendicular to the nuclear symmetry axis.

The phase $(-)^{I-\Sigma j_p}$ is thought of as a matrix when j_p (the angular momentum of the p^{th} particle relative to the potential) is not a constant of the motion. The normalization factor comes from the particular normalization of the rotational wave functions $\mathfrak{D}^I_{MK}(\theta_i)$, where θ_i refers to the Eulerian angles. The normalization is such that the wave functions represent unitary transformations from the coordinate system (x'', y'', z'') to the nuclear coordinate system (x', y', z').

The present paper consists of two main parts. In the first part a model is formulated for the interaction of the nucleons with the deformed nuclear field by introducing a single-particle Hamiltonian of a simple type, essentially containing a modified ellipsoidal oscillator potential and a spin-orbit term. A convenient representation, using the eigenvectors of an isotropic three-dimensional harmonic oscillator as a basic set, is then introduced. The calculated single-particle eigenvalues and eigenfunctions, obtained by means of an electronic digital computer, are arranged in tables and diagrams.

In the second part of the paper the applications of the single-particle states which have been calculated are discussed. First, we deal with the possibility of obtaining the total internal energy of the nucleus, the equilibrium deformation, and levels of particle excitation. Finally, expressions for the decoupling factor in rotational spectra, the magnetic moment, and the electromagnetic transition probabilities are given in terms of the particular wave-function representation chosen.

An analysis of empirical data (e. g. level spins, parities, magnetic moments, excitation spectra, and transition probabilities) compared with the results of the model is to be undertaken at a later date.* Already at this point, however, the general results of the calculations are published because of their wide range of application to different problems of nuclear physics.

II. Calculation of the Binding States in a Deformed Potential.

a. *Choice of field.*

To represent the interaction of one nucleon with the nuclear field we assume a single-particle Hamiltonian of the following

* Note added in proof: For preliminary results of such an analysis cf. B. Mottelson and S. G. Nilsson, Zs. f. Physik **141**, 217 (1955), and B. Mottelson and S. G. Nilsson, Phys. Rev. (in press).

form*, **. (For the sake of completeness we should really add a suffix p to all the quantities in this section, referring to the fact that they are single-particle quantities. However, we simplify the notation by dropping this index from the beginning.)

$$H = H_0 + C\,\bar{l}\cdot\bar{s} + D\,\bar{l}^2, \tag{2}$$

where

$$H_0 = -\frac{\hbar^2}{2M}\Delta' + \frac{M}{2}\left(\omega_x^2 x'^2 + \omega_y^2 y'^2 + \omega_z^2 z'^2\right), \tag{2 a}$$

where x', y', z' are the coordinates of a particle in a coordinate system fixed in the nucleus.

This means that an oscillator potential is first adopted for the sake of simplicity. To this is added the usual spin-orbit term. The $\bar{l}^2$-term then gives a correction to the oscillator potential especially at large distances (important for high l-values). This serves to depress the high angular momentum states. One might also say it has some of the features of the interpolation, between the square well and the oscillator potential, which is usually employed in the shell model. In the case of spherical symmetry, we must require that (2) and (2a) give the known sequence of single-particle levels considered in the shell model. This puts a strong limitation on the choice of C and D (see further the discussion on p. 15). In Fig. 2 one can compare the level scheme (with our parameter choice) for the spherical case, with the level spectrum proposed by KLINKENBERG***, which represents a compilation of empirical data interpreted on the basis of the shell model.

It is expected that many features of nuclear states obtained

* The author is indebted to Dr. A. BOHR, Dr. B. MOTTELSON, and Prof. I. WALLER for suggestions regarding the choice of a simple potential.

** Several authors have considered the motion of nucleons in deformed fields. E. FEENBERG and K. C. HAMMACK, Phys. Rev. **81**, 285 (1951), and S. GALLONE and C. SALVETTI, Il Nuovo Cimento (9) **8**, 970 (1951), consider an ellipsoidal square well, and D. PFIRSCH, Zs. f. Physik **132**, 409 (1952), treats the anisotropic harmonic oscillator, all using perturbation theory. S. GRANGER and R. D. SPENCE, Phys. Rev. **83**, 460 (1951), report that they have an exact solution for an infinitely deep spheroidal well, without any $\bar{l}\cdot\bar{s}$-term in the Hamiltonian, however. Finally, PFIRSCH, in the publication mentioned, and S. GALLONE and C. SALVETTI, Il Nuovo Cimento (9) **10**, 145 (1953), have studied the exact solutions of an anisotropic harmonic oscillator without spin-orbit force.

*** P. F. A. KLINKENBERG, Rev. Mod. Phys. **24**, 63 (1952).

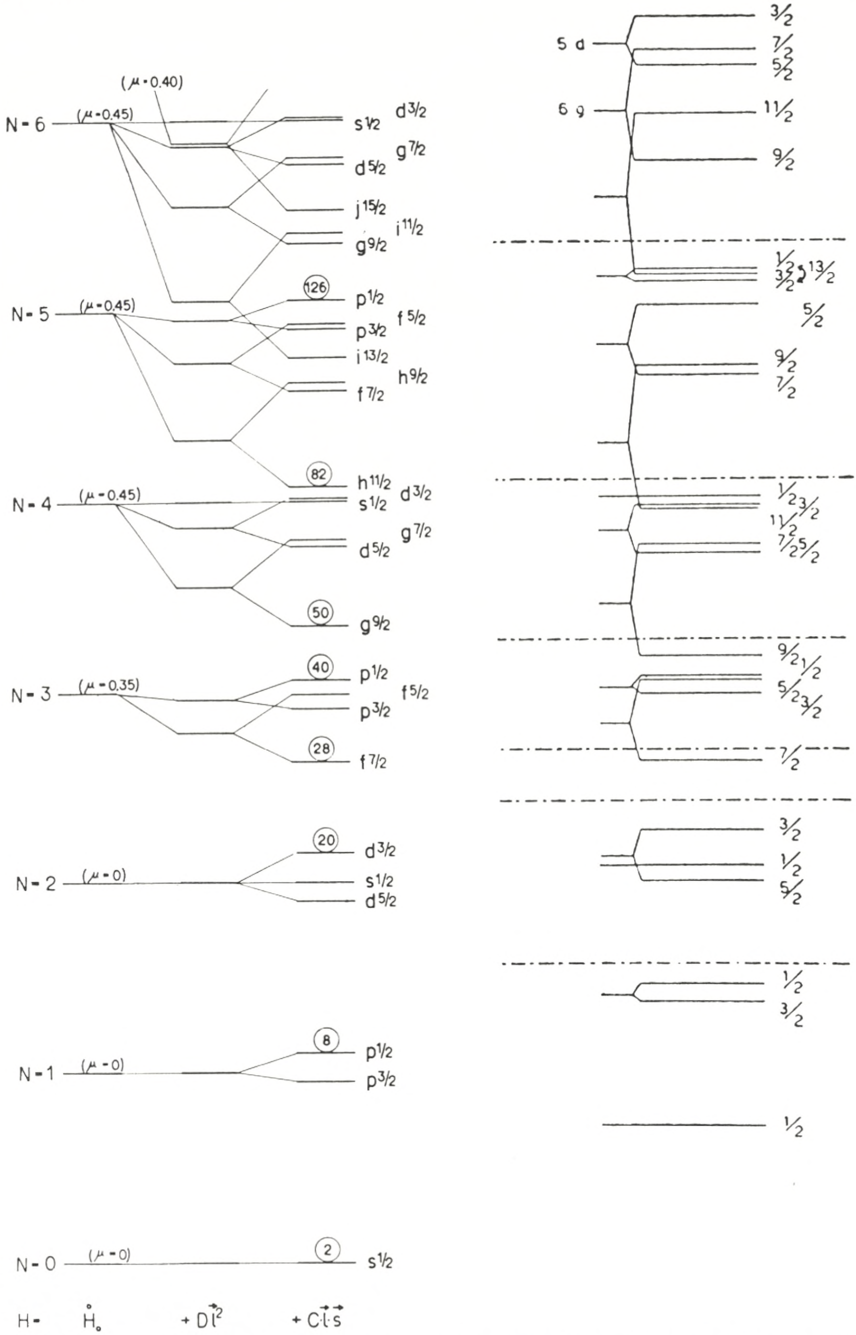

Fig. 2. *Level order for the spherical case compared with the shell model level order.*

Energy levels of the potential assumed in formula (2) for the spherical case ($\delta = 0$) are plotted to the left. The right part of the figure shows the level scheme proposed by P. Klinkenberg, which he has obtained from empirical data interpreted according to the shell model. The level scheme of Dr. Klinkenberg is reproduced by his kind permission from Reviews of Modern Physics.

from (2) are more general than the particular field employed, since the structure is especially determined by the angular properties, while the radial matrix elements alone reflect the detailed properties of the nuclear field.

We confine ourselves to the case of cylindrical symmetry and further introduce one single parameter of deformation δ

$$\omega_x^2 = \omega_0^2\left(1+\frac{2}{3}\delta\right) = \omega_y^2 \tag{3 a}$$

$$\omega_z^2 = \omega_0^2\left(1-\frac{4}{3}\delta\right). \tag{3 b}$$

Neglecting the $\bar{l}\cdot\bar{s}$- and $\bar{l}^2$-terms the problem is separable in x', y', z'. In this case a change, e. g., of ω_x only changes the scale of the wave function along the x'-axis. As the scale is proportional to $\frac{1}{\sqrt{\omega_x}}$, the condition of constant volume of the nucleus leads to

$$\omega_x \omega_y \omega_z = \text{const.}$$

Keeping this condition in the general case together with (3a) and (3b), ω_0 has to depend on δ in the following way

$$\omega_0(\delta) = \mathring{\omega}_0\left(1-\frac{4}{3}\delta^2-\frac{16}{27}\delta^3\right)^{-1/6}. \tag{4}$$

$\mathring{\omega}_0$ is the value of $\omega_0(\delta)$ for $\delta = 0$. It turns out that δ is related to the quantity β, used in the papers by A. Bohr and B. Mottelson, to first-order as*

$$\delta \simeq \frac{3}{2}\sqrt{\frac{5}{4\pi}}\beta \simeq 0.95\,\beta. \tag{5}$$

We introduce new coordinates

$$x = \sqrt{\frac{M\omega_0}{\hbar}}\,x' \quad \text{etc.,} \tag{6}$$

and split H_0 into a spherically symmetric term $\mathring{H}_0$ and a term H_δ representing the coupling of the particle to the axis of the deformation

* Cf. (16) and BM (V. 7).

$$H_0 = \mathring{H}_0 + H_\delta, \tag{7}$$

where

$$\mathring{H}_0 = \hbar\omega_0 \frac{1}{2} [-\Delta + r^2] \tag{7a}$$

$$H_\delta = -\delta\hbar\omega_0 \frac{4}{3} \sqrt{\frac{\pi}{5}} r^2 Y_{20}.\ * \tag{7b}$$

b. *Choice of representation.*

A representation is chosen with $\mathring{H}_0$ diagonal, together with $\bar{l}^2$, l_z, and s_z, which all commute with $\mathring{H}_0$. The corresponding quantum numbers are denoted l, Λ and Σ.

None of the above operators commute with the total Hamiltonian. A commuting operator, however, is $j_z = l_z + s_z$. We denote the corresponding quantum number by Ω. For the states corresponding to a given Ω, the vectors $|Nl\Lambda\Sigma\rangle$ with $\Lambda + \Sigma = \Omega$ are used as basic vectors. The quantum number N represents the total number of oscillator quanta. One has

$$\mathring{H}_0 |Nl\Lambda\Sigma\rangle = \left(N + \frac{3}{2}\right)\hbar\omega_0 |Nl\Lambda\Sigma\rangle.$$

In configuration space representation the basic vector looks like

$$\langle \bar{r} | Nl\Lambda\Sigma\rangle \sim r^l e^{-\frac{1}{2}r^2} F\left(-n, l + \frac{3}{2}, r^2\right) Y_{l\Lambda} f_{s\Sigma}, \tag{8}$$

where the relation

$$2n + l = N \tag{8a}$$

defines n. Further $F\left(-n, l + \frac{3}{2}, r^2\right)$ is the confluent hypergeometric function.

In this representation the different parts of the total Hamil-

* We assume here and throughout this paper that the phases of the spherical harmonics are chosen in accordance with E. U. Condon and G. H. Shortley, The Theory of Atomic Spectra, Camb. Univ. Press, London (1935).

tonian have very simple matrix elements; in particular are $\bar{l}^2$ and $\mathring{H}_0$ diagonal in this representation.

The matrix elements $\langle l'\Lambda'\Sigma' \,|\, \bar{l}\cdot\bar{s} \,|\, l\Lambda\Sigma \rangle$ have the following selection rules

$$l = l', \; \Lambda = \begin{cases} \Lambda' \\ \Lambda' \pm 1 \end{cases}, \quad \Sigma = \begin{cases} \Sigma' \pm 1 \\ \Sigma' \end{cases}, \; \Lambda + \Sigma = \Lambda' + \Sigma'.$$

We write down, for completeness, the non-vanishing elements

$$\langle l \Lambda \pm 1 \mp \,|\, \bar{l}\cdot\bar{s} \,|\, l\Lambda \pm \rangle = \frac{1}{2}\sqrt{(l \mp \Lambda)\,(l \pm \Lambda + 1)} \qquad (9\,\mathrm{a})$$

$$\langle l\Lambda \pm \,|\, \bar{l}\cdot\bar{s} \,|\, l\Lambda \pm \rangle = \pm \frac{1}{2}\Lambda, \qquad (9\,\mathrm{b})$$

denoting $\Sigma = +\frac{1}{2}$ and $\Sigma = -\frac{1}{2}$ simply by $+$ and $-$, respectively, in the vectors.

The only part of the Hamiltonian not immediately given is H_δ, which is proportional to $r^2 Y_{20}$. It is easy to show that

$$\langle l'\Lambda' \,|\, Y_{20} \,|\, l\Lambda \rangle = \sqrt{\frac{5}{4\pi}}\sqrt{\frac{2l+1}{2l'+1}}\langle l\,2\,\Lambda\,0 \,|\, l\,2\,l'\Lambda' \rangle \langle l\,2\,0\,0 \,|\, l\,2\,l'\,0 \rangle \qquad (10)$$

in the Condon-Shortley notation for Clebsch-Gordon coefficients.

Matrix elements of r^2 are calculated most easily with the help of recursion formulae for confluent hypergeometric functions. The general matrix element for r^λ is given later in (41), but the simplified expressions for $\lambda = 2$ are given here:

$$\langle Nl \,|\, r^2 \,|\, Nl \rangle = N + \frac{3}{2} \qquad (11\,\mathrm{a})$$

$$\langle Nl-2 \,|\, r^2 \,|\, Nl \rangle = 2\sqrt{(n+1)\left(n+l+\frac{1}{2}\right)} \qquad (11\,\mathrm{b})$$

$$\langle N-2\,l \,|\, r^2 \,|\, Nl \rangle = \sqrt{n\left(n+l+\frac{1}{2}\right)} \qquad (11\,\mathrm{c})$$

$$\langle N-2\ l-2 \mid r^2 \mid Nl \rangle = \sqrt{\left(n+l+\frac{1}{2}\right)\left(n+l-\frac{1}{2}\right)} \quad (11\,\mathrm{d})$$

$$\langle N-2\ l+2 \mid r^2 \mid Nl \rangle = \sqrt{n(n-1)}. \quad (11\,\mathrm{e})$$

The selection rules for $r^2 Y_{20}$ are

$$\Lambda = \Lambda', \quad \Sigma = \Sigma', \quad l = \begin{cases} l' \\ l' \pm 2 \end{cases}, \quad N = \begin{cases} N' \\ N' \pm 2 \end{cases}.$$

The selection rules for $r^2 Y_{20}$ imply that there is a coupling between states with different N (the difference in N being an even number). The approximation is made, however, that this coupling is neglected. Levels belonging to, e. g., the N-shell and the $(N+2)$-shell are on the average separated by an energy of $2\,\hbar\omega_0$, which, for most values of the parameters, is much larger than the corresponding non-diagonal coupling energies.

In fact, it can be shown by changing the representation slightly, that these couplings between shells of different N can be accounted for by a small change in the interpretation of the parameters δ, ω_0 etc., and a small modification of the $\bar{l}\cdot\bar{s}$- and $\bar{l}^2$-terms (cf. Appendix A).

Non-vanishing matrix elements of H are thus considered only between base vectors $\mid Nl\Lambda\Sigma\rangle$ belonging to the same N and Ω.

A note should be made at this point that there are a few cases when levels of the same spin and parity (but belonging to N-shells with N different by two) cross each other within the range of the parameter η considered. Fig. 5 shows two such crossings between levels of the $N=4$ and the $N=6$ shells. One crossing occurs between the $\Omega = 1/2$ levels # 51 and # 60 and the other between the $\Omega = 3/2$ levels # 42 and # 57. (Concerning the labelling of the levels, see p. 19.) These crossings are removed when account is taken of the neglected coupling terms between the N- and the $(N+2)$-shells.* The coupling terms between the crossing levels are calculated in Table II.

* The corresponding energy levels of the Hamiltonian H_t, considered in Appendix A, actually cross in the exact treatment for a deformation different from zero. With that form of the Hamiltonian (H_t) there is, however, associated an additional degeneracy compared to the case considered here.

c. Details of calculations.

From (2) and (7) we have

$$H = \mathring{H}_0 + H_\delta + C\bar{l}\cdot\bar{s} + D\bar{l}^2 .$$

As $\mathring{H}_0$ is diagonal in the representation chosen, and its matrix elements are all equal for a constant N, it is advantageous to bring $\mathring{H}_0$ out of the matrix H and only consider $H - \mathring{H}_0$.

We introduce new parameters μ and $\varkappa$ instead of C and D

$$\varkappa = -\frac{1}{2}\frac{C}{\hbar\mathring{\omega}_0} \tag{12a}$$

$$\mu = \frac{2D}{C} . \tag{12b}$$

Further we introduce a new $\varkappa$-dependent deformation parameter

$$\eta = \frac{\delta}{\varkappa}\frac{\omega_0(\delta)}{\mathring{\omega}_0} = \frac{\delta}{\varkappa}\left[1 - \frac{4}{3}\delta^2 - \frac{16}{27}\delta^3\right]^{-1/6} . \tag{12c}$$

We can write

$$H_\delta = \delta\hbar\omega_0 U = \varkappa\hbar\mathring{\omega}_0\cdot\eta\cdot U ,$$

where

$$U = -\frac{4}{3}\sqrt{\frac{\pi}{5}}\, r^2 Y_{20} \tag{12d}$$

which does not depend on δ.

It is then convenient to write

$$H - \mathring{H}_0 = \varkappa\hbar\mathring{\omega}_0 R , \tag{12e}$$

where

$$R = \eta U - 2\,\bar{l}\cdot\bar{s} - \mu\bar{l}^2 \tag{12f}$$

is an operator that depends only on two parameters, η and μ.

The final calculations now consist of an exact diagonalization of the (dimensionless) matrix R in the representation chosen. R is treated as a function of the deformation parameter η, and it is diagonalized for a sequence of η-values (cf. below). The only other parameter that enters R is μ, which is independent of the deformation. The choice of μ is discussed below.

From the diagonalization of R, or rather its submatrices belonging to certain N and Ω, we obtain the eigenvalues $r_{\alpha}^{N\Omega}(\eta)$. (Here α numbers the different eigenvalues of the matrix.) The corresponding energy eigenvalues of the total H are then given as

$$E_{\alpha}^{N\Omega} = \left(N_{\alpha} + \frac{3}{2}\right)\hbar\,\omega_0\,(\delta) + \varkappa\hbar\,\mathring{\omega}_0\, r_{\alpha}^{N\Omega}. \tag{13}$$

Let us denote the corresponding eigenvector $|N\Omega\alpha\rangle$. Its configuration space representation is denoted by χ_{Ω_p} in the Introduction.

The values of $\varkappa$ and μ are chosen, as mentioned earlier, in such a way that for $\delta = 0$ the sequence of levels of the shell model are reproduced. Of course we are free to let both $\varkappa$ and μ vary from shell to shell, i. e. vary with N.

The parameter μ determines the sequence of levels within the group of states belonging to a particular N by depressing (for $\mu > 0$) the levels corresponding to higher l-values. The total energy spread of levels belonging to the same N-shell is determined primarily by the parameter $\varkappa$. In the numerical calculations we have assigned values of μ for each N-shell so as to reproduce (for $\delta = 0$) the assumed sequence of shell model levels as well as possible.

In the numerical calculations μ is chosen in the following manner

$N = 0,\ 1,\ 2$	$\mu = 0$
$N = 3$	$\mu = 0.35\ (0,\ 0.50)$
$N = 4$	$\mu = 0.45\ (0.55)$
$N = 5,\ 6$	$\mu = 0.45$
$N = 7$	$\mu = 0.40.$

In general, this choice of μ means that in the lower N-shells we use a pure oscillator and for higher N-values we approach more to a square well (cf. Fig. 2).

In order to examine the sensitivity of our results to the particular choice of μ, we have performed calculations for the shell with $N = 3$ employing a sequence of different μ-values. The resulting level spectra are plotted as functions of the deformation parameter η in Figs. 3a, b, c. It is seen that, even for rather different choices of the level spectrum in the spherical potential ($\eta = 0$), the results become quite similar for large η.

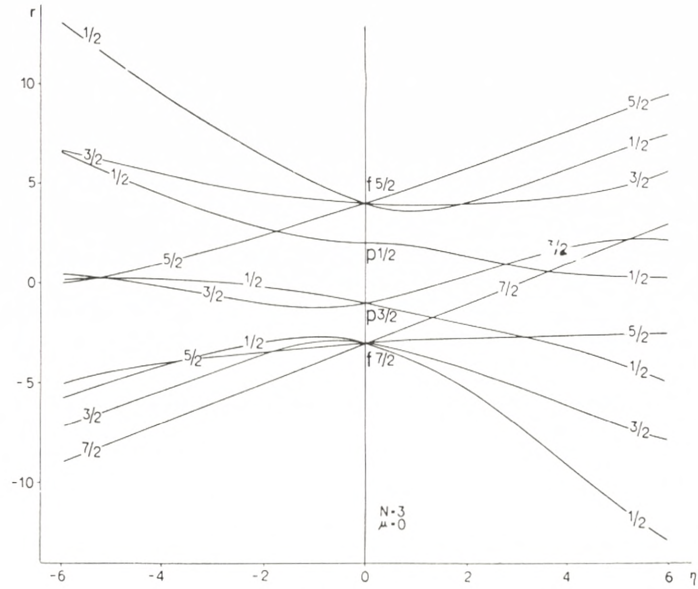
r
10
5
0
-5
-10
-6
-4
-2
0
2
4
6
η
f 5/2
p1/2
p3/2
f 7/2
N = 3
μ = 0

Fig. 3 a.

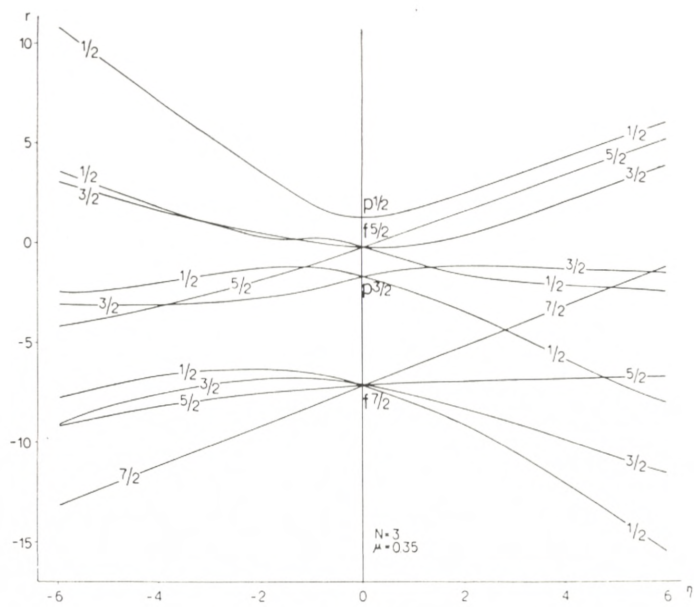
r
10
5
0
-5
-10
-15
-6
-4
-2
0
2
4
6
η
p1/2
f5/2
p3/2
f7/2
N = 3
μ = 0.35

Fig. 3 b.

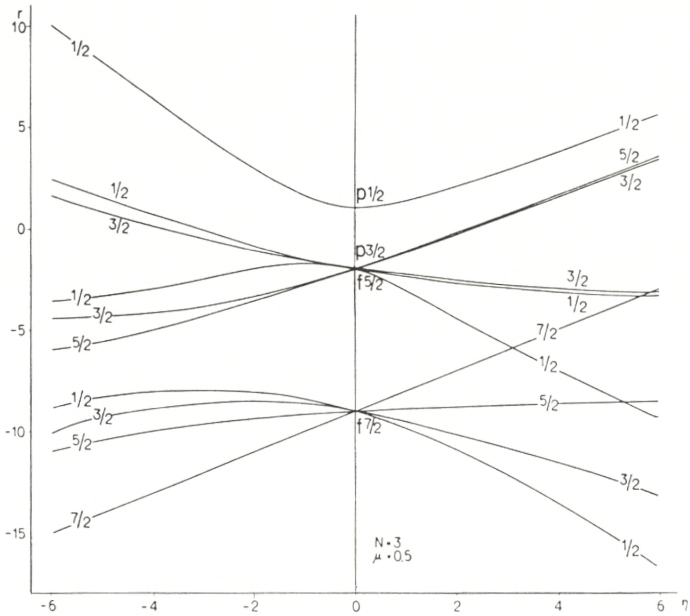

Fig. 3 c.

Fig. 3 a, b, c. The influence of the choice of $\bar{l}^{2}$*-admixture in the potential on the energy eigenvalues.*

Eigenvalues $r(\eta)$ corresponding to $N = 3$ are depicted as functions of the deformation parameter η, with three different choices of μ, the parameter of $\bar{l}^{2}$-admixture in the assumed potential. The connection between the eigenvalues $r(\eta)$ and the level energy E is given by (13). One may notice that for large η-values (large deformations) the level order within the $N = 3$ groups of levels is rather independent of μ.

Finally, as regards the choice of $\varkappa$, one sees from (13) that the level spread within each N-shell is directly proportional to $\varkappa$. As the N-shells overlap for a larger number of nucleons, $\varkappa$ has to be chosen within certain limits to reproduce for $\delta = 0$ the level order of the shell model. The arbitrariness in the choice of $\varkappa$ is important particularly for $N \leq 3$.

In the final plot (Fig. 5) the value of $\varkappa$ is taken to be 0.05 for all levels. It is, however, easy to modify the plotting and use the result of the calculation for another $\varkappa$-value. This will mean two things: a) the same value of η now corresponds to another deformation δ according to (12c), b) the second term in (13) is changed.

A reasonable value for $\overset{\circ}{\omega}_0$ may be obtained by taking the mean value of r'^2 for all the nucleons to be equal to $\frac{3}{5}\cdot(1.2\cdot 10^{-13}A^{\frac{1}{3}})^2\,\mathrm{cm}^2$, which gives $\hbar\overset{\circ}{\omega}_0 \simeq 41\,A^{-\frac{1}{3}}$ MeV.*

Thus for $A \simeq 100$ one has $\hbar\overset{\circ}{\omega}_0 \simeq 8.8$ MeV. This choice of $\hbar\overset{\circ}{\omega}_0$ and the choice of $\varkappa$ made in the plot ($\varkappa = 0{,}05$) give, e. g., a spin-orbit splitting between $g_{9/2}$ and $g_{7/2}$ of 4.0 MeV.

In the calculation, submatrices of R were diagonalized up to and including $N = 6$. The largest, which corresponds to $N = 6$, $\Omega = \frac{1}{2}$, is then a 7×7 matrix. The calculation is repeated for six values of η ($\eta = -6, -4, -2, 2, 4, 6$). Matrices of order 3×3 and higher were treated with the help of the digital computing machine BESK in Stockholm. A method due to Jacobi was used in the machine calculations for matrix diagonalization.

Finally, the case $\delta = 0$ (or $\eta = 0$) corresponds to spherical symmetry and is already worked out. One obtains the behaviour of the levels in the vicinity of $\delta = 0$ by introducing an $|\,Nlj\Omega\,\rangle$-representation. Here the Hamiltonian is diagonal except for H_δ, which can be treated as a perturbation for small δ.

d. *Arrangements of tables and main diagram.*

Table I gives the eigenvalues $r(\eta)$ and the corresponding eigenfunctions as a sequence of coefficients $A_{l\,\Omega-1/2}$ and $A_{l\,\Omega+1/2}$, defined by

$$|\,N\Omega\alpha\,\rangle = \sum_l \left\{A_{l\,\Omega-1/2}\,|\,Nl\,(\Omega-1/2)+\rangle + A_{l\,\Omega+1/2}\,|\,Nl\,(\Omega+1/2)-\rangle\right\}, \quad (14)$$

where the normalization of $A_{l\Lambda}$ is discussed below. (If we write (14) with coefficients $a_{l\Lambda}$, we assume a normalization $\sum_{l\Lambda} a^2_{l\Lambda} = 1$.) The basic vectors $|\,Nl\,(\Omega-1/2)+\rangle$ and $|\,Nl\,(\Omega+1/2)-\rangle$ are given above each separate table.

Consider, as an example, $N = 5$, $\Omega = \frac{5}{2}$. Above the table are written the base vectors $|\,552+\rangle$, $|\,532+\rangle$, $|\,553-\rangle$, and

* An estimate for the harmonic oscillator potential in accordance with the Thomas-Fermi statistical model agrees with what one obtains by using the wave functions of the individual nucleons.

$|533-\rangle$. The eigenvalues $r_1, r_2, \ldots, r_4$ are listed for each value of η. Below each of them there are four numbers, which are the coefficients $A_{52}, A_{32}, \ldots$ with a normalization such that the first listed coefficient equals 1. Take, e. g., $\eta = -4$. The largest eigenvalue is -2.676. To this corresponds the eigenfunction $1.000\ |552+\rangle + 1.355\,|532+\rangle - 1.030\ |553-\rangle - 1.074\ |533-\rangle$.

The numbers to the left in Tables I are referring to the curves in Fig. 5. In some instances these numbers are missing. This means that the level in question lies outside the range of the energy scale used in the diagram. Curves coming into the drawing from above left are labelled by letters.

Fig. 5 shows the energy eigenvalues E_α given by (13) as functions of the deformation parameters η or δ, to which latter η is related by (12 c). The scales for η and δ are shown at the bottom of the drawing. The calculated points corresponding to $\eta = -6, -4, -2, 0, 2, 4, 6$ are fitted by curves with the further requirement of a given slope at $\eta = 0$ (determined from perturbation calculations, cf. above). The curves are labelled by the Ω-number and the parity sign. The energy scale is $\hbar\omega_0(\delta)$; this δ-dependent unit is chosen rather than the constant unit $\hbar\mathring{\omega}_0$ to simplify the drawing. What is plotted is $\frac{E_\alpha}{\hbar\omega_0} = \left(N_\alpha + \frac{3}{2}\right) + \frac{\mathring{\omega}_0}{\omega_0(\delta)} \cdot r_\alpha$. Notice, further, that the true energy scale is different for nuclei with different A, as $\hbar\omega_0$ may be assumed to vary with A as $A^{-1/3}$ (cf. p. 18).

The bottom level, corresponding to $\Omega = \frac{1}{2}+$, which is a pure $|000+\rangle$-state, is left out in Fig. 5 in order to save space.

Of the $N = 7$ states only the $\Omega = \frac{15}{2}$ state has been calculated, and thus one must expect additional levels in the diagram for energies above, say, $6.6\ \hbar\omega_0$.

Added in proof: In the analysis of the empirical level spectra (B. Mottelson and S. G. Nilsson, Phys. Rev., in press) it has been found that an improved fit for the protons in the $N = 4$ shell is obtained by increasing slightly the value of μ. The calculations have therefore been performed also for $\mu = 0.55$. The results are given in Table 1 b. (The eigenvectors are normalized such that $\sum_{l\Lambda} a_{l\Lambda}^2 = 1$.) The corresponding energy level diagram is shown in the above reference. In this diagram $\varkappa$ is chosen $= 0.0613$, compared to 0.05 in Fig. 5.

e. *Discussion of the main level diagram.*

Many of the features of the level diagram in Fig. 5 can be understood from simple considerations.

Thus, in the neighbourhood of spherical shape, the states can be labelled by the l and j quantum numbers. The degeneracies corresponding to $\delta = 0$ are removed by the surface coupling term in such a manner that for a positive δ the energies increase with increasing Ω. For negative δ the level order is the opposite.

With increasing deformation, the states of different j and l (but the same Ω and parity) are coupled together, and for intermediate deformations the situation may be rather complex, as seen from the peculiar variations with δ of some of the energy levels.

For sufficiently large deformations, the situation again simplifies since for this case one may consider as a zeroth approximation the levels of a pure (anisotropic) harmonic oscillator potential and treat the $\bar{l}\cdot\bar{s}$- and $\bar{l}^2$-terms as a perturbation. In this limit the states may be labelled by the quantum numbers N, n_z (number of oscillator quanta along the z'-axis), Λ and Σ.

The quantum numbers appropriate at large deformations can easily be assigned to the energy levels in Fig. 5 by noting the following rules. For the levels in the shell N the lowest state of $\Omega = 1/2$, assuming positive deformation, has $n_z = N$, the next has $n_z = N - 1$ etc., so that the highest $\Omega = 1/2$ level has $n_z = 0$. Similarly for $\Omega = 3/2$ the lowest level has $n_z = N - 1$, the next $n_z = N - 2$ etc. After the n_z-values have been assigned, the Λ-values can be simply obtained by noting that Λ is even or odd according to whether $(N - n_z)$ is even or odd. Since Ω is known, this determines Λ and Σ uniquely.

As an example, it is in this way found that in the $N = 5$ shell the levels corresponding to $n_z = 0$ for large positive deformations are the ones labelled 28, 48, 40, 70, 61, and A. It is also seen that all these levels tend to become parallel and to increase steeply with the deformation, corresponding to the fact that for these levels the oscillations are in the plane of the small nuclear axes. At the same time it is apparent that the $\bar{l}\cdot\bar{s}$- and $\bar{l}^2$-terms in the Hamiltonian, which are responsible for the spin-orbit

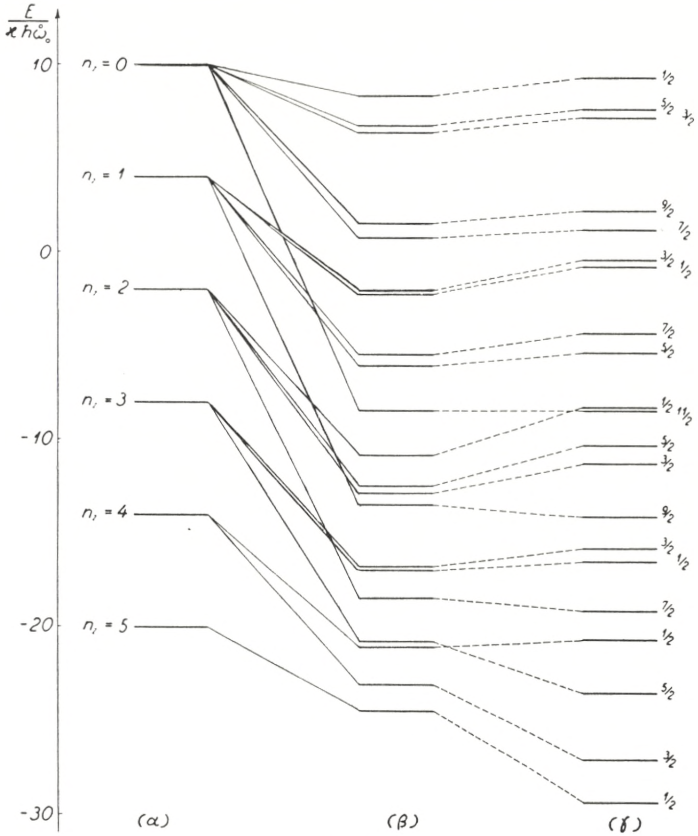

Fig. 4 a.

Figs. 4 a and 4 b. Comparison of a perturbation treatment with the exact calculations.

Energy levels for the $N = 5$ shell are plotted in units of $\varkappa\hbar\mathring{\omega}_0$ in Fig. 4a for a deformation defined by $\eta = 6$ and in Fig. 4b for a deformation $\eta = -6$. The group of levels (α) represent the eigenvalues E_0 of the pure oscillator potential H_0, (β) includes first order perturbation terms of $\bar{l}\cdot\bar{s}$ and $\bar{l}^2$, while (γ) shows the energy eigenvalues obtained by the exact machine calculations.

splitting, have still a very appreciable influence on the level order.

In Figs. 4a and 4b the energy levels obtained by treating the $\bar{l}\cdot\bar{s}$- and $\bar{l}^2$-terms as a perturbation are compared with the exact level spectrum for the $N = 5$ shell in the case of the largest deformation considered in Fig. 5. To the left in Figs. 4a and 4b

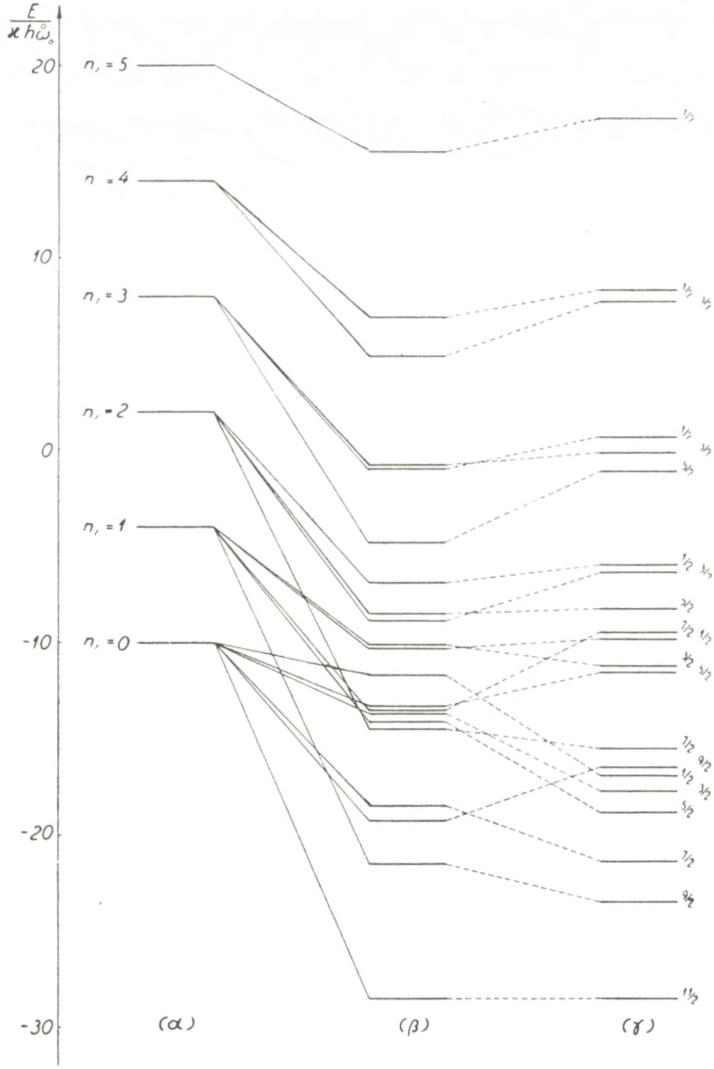

Fig. 4 b.

are the pure oscillator levels (α), while the levels (β) include the diagonal values of the $\bar{l}\cdot\bar{s}$- and $\bar{l}^2$-terms, which are calculated in Appendix B. The comparison with the numerically calculated levels (γ) shows that such a first order perturbation calculation, for the very large deformations in question, reproduces the main trends of the level order, even though there are still a number of significant differences between the level spectra (β) and (γ). This is especially the case for negative deformations (cf. Fig. 4 b).

III. Examples of Applications of Tables and Diagrams.

We consider below a number of the nuclear properties which may be treated by means of the calculation given above. It should be remembered that the essential condition, underlying all of the work in the present paper, is that the nuclear deformation is essentially larger than the fluctuations. This condition is found to be satisfied only for configurations far removed from the closed shells.

a. *Calculation of total energy and equilibrium deformation.*

The total energy of the nucleus is not the sum of the energies for each individual particle because, in that case, two-particle interactions would be counted twice, three-particle interactions three times, etc.

The expression to be used thus depends on which kind of interaction is postulated. Assuming only two-body forces, the Hamiltonian for the i:th particle is

$$H_i = T_i + V_i = T_i + \sum_{j\,(j \neq i)} V_{ij}.$$

The Hamiltonian for the total nucleus, however, should be

$$\mathfrak{H} = \sum_i T_i + \frac{1}{2} \sum_{\substack{i,j \\ (i \neq j)}} V_{ij} = \frac{1}{2} \sum_i H_i + \frac{1}{2} \sum_i T_i. \tag{15}$$

The total wave function of all the nucleons, describing their motion relative to the deformed potential, is the product of the wave functions for each occupied particle state, appropriately antisymmetrized. To find the total energy $\mathfrak{E}(\delta)$ we then have to find the expectation value of $\mathfrak{H}$ with respect to the calculated single-particle wave functions.

The equilibrium deformation δ_{eq} is now given by $\left(\frac{\partial \mathfrak{E}(\delta)}{\partial \delta}\right)_{\delta_{\text{eq}}} = 0$, which gives the minimum total energy $\mathfrak{E}_{\text{min}}$. The energy values are calculated with good accuracy for seven points on all the levels. As the levels cross, different combinations of levels will give the lowest total energy within different ranges of δ (or η).

For each single combination of levels the energy minimum is to be found, e. g., from an interpolation formula utilizing several or possibly all of the calculated points corresponding to this particular level combination. The lowest minimum gives the ground state. Other minima, corresponding to other level combinations, give particle levels of the excitation spectrum.

In this connection it should be emphasized that the total nuclear excitation spectrum will have three distinct modes. On each particle level, characterized by Ω, there will be superimposed a vibrational band, and furthermore on each level (including the vibrational levels) a rotational band. The level distance for heavy nuclei is for the particle spectrum of order 100 KeV (see Fig. 5), for the vibrational spectrum of the order of a few MeV. Finally, the rotational energies depend on the nuclear deformation, but for heavy nuclei and large deformations they are much smaller than the vibrational energies.

To calculate the equilibrium deformation in the prescribed way it turns out to be essential to take the couplings between different N-shells into account. This can be done, as pointed out on p. 13, by reinterpreting the machine calculations as performed in a slightly different representation, accompanied by a small change in the definition of the parameters (cf. Appendix A).

This coupling causes a slight repression of all the energy levels without affecting the level order. It thus amounts to a change in the whole energy scale (cf. (A4), (4), and (13)). Furthermore the scale factor is dependent on deformation.

The effect is important in decreasing the effective restoring force of the nucleus against deformation. The energy minimum is thus shifted towards larger deformations. An approximate expression for the total energy $\mathfrak{E}$, taking these effects into account, is given in Appendix C.

It should be emphasized that the determination of δ_{eq} involves a number of simplifying approximations. Apart from the assumption regarding the shape of the nuclear potential and the two-body character of the interactions, we have neglected the effect of residual interactions between the nucleons not included in the average potential (as, e. g., the pairing energy terms).

However, in the application of the model an independent estimate of δ is obtained from the empirically determined quadrupole moment. Assuming a charge distribution in accordance with the Thomas-Fermi statistical model applied to the oscillator potential, one obtains to second order in δ

$$Q_0 \simeq 0.8 \cdot Z \cdot R_Z^2 \cdot \delta \cdot \left(1 + \frac{2}{3}\delta\right), \tag{16}$$

where R_Z is to be taken equal to the radius of charge of the nucleus or $R_Z \simeq 1.2 \cdot 10^{-13} \cdot A^{1/3}$ cm. In obtaining this result the convention has been employed to put the mean value of r'^2 for all the protons (cf. p. 18) equal to $3/5\, R_Z^2$.

The relation between the measured quadrupole moment, denoted by Q_s, and Q_0 is given by*

$$Q_s = \frac{3\,K^2 - I\,(I+1)}{(I+1)(2\,I+3)}\,Q_0. \tag{17}$$

As regards the particle levels of the excitation spectrum, one cannot expect to obtain the exact level order and even less the correct energy differences between the levels. The diagram should tell, however, which level spins and parities are likely to appear in the lowest states of the spectrum.

b. *Determination of ground state spin and decoupling factor.*

The component of angular momentum along the axis of deformation Ω_p is a constant of the motion for each particle. Ω_p is given for each of the energy states drawn in Fig. 5. In the strong coupling limit the total Ω equals $\sum_p \Omega_p$. Each energy state is degenerate corresponding to $\pm\Omega$. If we have two groups a and b of equivalent particles — neutrons and protons — the particles of each group fill pairwise in the levels independently of the other group. If the number of particles in the group a is even, $\Omega_a = 0$, if odd, then Ω_a equals the Ω_p of the last unpaired particle.

If one of the groups is even, the case is simple enough for the ground state, Ω equals Ω_b, if b is the odd group. If both a and b are odd, the states with $\Omega = \left|\Omega_a \pm \Omega_b\right|$ are degenerate

* BM (V. 6).

in first order. The diagonal contribution of n-p-forces and the rotational energy decide which Ω corresponds to the ground state.

It turns out that always the ground state spin of the nucleus $I_0 = \Omega = K$, except when $\Omega = \frac{1}{2}$, in which case the ground state spin I_0 is given from Table III once the decoupling factor a is determined.* (See below.)

The decoupling factor a appears in the expression for the rotational energy for odd-A nuclei with $\Omega = \frac{1}{2}$**

$$E_{\text{rot}} = \frac{\hbar^2}{2\mathfrak{J}}\left[I\,(I+1) + a\,(-)^{I+1/2}\left(I+\frac{1}{2}\right)\right] \qquad (18)$$

and is thus experimentally measurable. In the j-Ω-representation, with quantum numbers l, s, j, Ω, where χ_Ω is written $\sum_j c_j \langle \bar{r} \mid Nlj\Omega \rangle$,

$$a = \sum_j (-)^{j-1/2}\left(j+\frac{1}{2}\right) \mid c_j \mid^2,$$

and it can be transformed to the l-Λ-representation with quantum numbers l, s, Λ, Σ, where $\chi_\Omega = \sum_{\substack{l\Lambda \\ \Lambda+\Sigma=\Omega}} a_{l\Lambda} \langle \bar{r} \mid Nl\,(s)\,\Lambda\Sigma \rangle$, by means of the relations

$$c_j = \sum_{\Lambda\Sigma} \langle\, l\,\frac{1}{2}\,\Lambda\Sigma \mid l\,\frac{1}{2}\,j\,\Omega \,\rangle\; a_{l\Lambda}.$$

In the l-Λ-representation then

$$a = (-)^l \sum_l \left(a_{l0}^2 + 2\sqrt{l\,(l+1)}\; a_{l0}\, a_{l1}\right), \qquad (19)$$

where $(-)^l$ is the parity of the state in question, and where the coefficients $a_{l\Lambda}$ are, as before, the representatives of the particle wave function χ_Ω in the $\mid Nl\Lambda\Sigma \rangle$-representation. The

* Cf. BM p. 30. Table III is based on BM (II.24). I_0 is determined as the half integer spin I which gives the minimum rotational energy W_{rot}.

** See A. Bohr and B. Mottelson "Collective Nuclear Motion and the Unified Model", Chapter 17 of "Beta and Gamma Ray Spectroscopy" edited by K. Siegbahn, North Holland Publishing Co. (1954).

values of $a_{l\Lambda}$ for the calculated eigenstates are the same as the coefficients listed in Tables I, apart from a different normalization (cf. p. 18).

c. Determination of magnetic moments.

We next consider the magnetic moment for an odd-A nucleus in which all the particles except the last one fill the different orbits in pairs. The generalization to configurations in which several particles move in unpaired orbits is straightforward.

By definition, the magnetic moment expressed in units of the nuclear magneton is

$$\mu = \frac{\langle \bar{\mu}^{\text{op}} \cdot \bar{I} \rangle}{I+1},$$

where

$$\bar{\mu}^{\text{op}} = g_s \bar{s} + g_l \bar{l} + g_R \bar{R},$$

and $\bar{R}$ is the angular momentum of the surface.

Using $\bar{j} = \bar{l} + \bar{s}$ and $\bar{j} + \bar{R} = \bar{I}$ we can write μ as

$$\mu = \frac{1}{I+1} \left[(g_s - g_l) \langle \bar{s} \cdot \bar{I} \rangle + (g_l - g_R) \langle \bar{j} \cdot \bar{I} \rangle + g_R \langle \bar{I}^2 \rangle \right]. \qquad (20)$$

Here $\langle \bar{j} \cdot \bar{I} \rangle$ is given in the j-Ω-representation of BM.* It can be written

$$\langle \bar{j} \cdot \bar{I} \rangle = \Omega K + \frac{1}{2}\left(I + \frac{1}{2}\right) a \, (-)^{I-1/2} \delta_{\Omega,\,1/2} \delta_{K,\,1/2}, \qquad (21)$$

where the l-Λ-representation of a is given directly from (19).

For $\langle \bar{s} \cdot \bar{I} \rangle$ it is more convenient to use the l-Λ-representation from the beginning. The part of the wave function (1) which is written $(-)^{I-j} \chi_{-\Omega}$ has the meaning $\sum_j (-)^{I-j} c_j \chi^j_{-\Omega}$. In the l-Λ-representation then

$$\sum_j (-)^{I-j} c_j \chi^j_{-\Omega} = (-)^{I-1/2+l} \sum_{l \Lambda \Sigma} a_{l\Lambda} \langle \bar{r} \,|\, N l \, -\Lambda \, -\Sigma \rangle. \qquad (22)$$

One obtains

* BM II.18.

$$\langle \bar{s} \cdot \bar{I} \rangle = \frac{K}{2} \sum_l (a_{l+}^2 - a_{l-}^2) + \frac{1}{2}\left(I + \frac{1}{2}\right)(-)^{I-1/2+l} \sum_l a_{l0}^2 \, \delta_{\Omega,\,1/2}\, \delta_{K,\,1/2}. \quad (23)$$

For the case $\Omega = K = I = \frac{1}{2}$, (23) simplifies to

$$\langle \bar{s} \cdot \bar{I} \rangle = \frac{1}{2} \sum_l a_{l0}^2 \, [1 + (-)^l] - \frac{1}{4} \quad (23\,\mathrm{a})$$

(where we have utilized $\sum_l (a_{l0}^2 + a_{l1}^2) = 1$). Specializing further to a state of odd parity,

$$\langle \bar{s} \cdot \bar{I} \rangle = -\frac{1}{4}. \quad (23\,\mathrm{b})$$

Turning now to μ, we first consider the case $\Omega \neq \frac{1}{2}$. From (20), (21), and (23) one obtains

$$\left.\begin{aligned} \mu = \frac{1}{I+1} \Big\{ & (g_s - g_l) \frac{1}{2} K \sum_l (a_{l\,\Omega-1/2}^2 - a_{l\,\Omega+1/2}^2) \\ & + (g_l - g_R)\, \Omega K + g_R\, I\,(I+1) \Big\}, \end{aligned}\right\} \quad (24)$$

which for $I = \Omega = K$ $\left(\text{still} \neq \frac{1}{2}\right)$ simplifies to

$$\mu = \frac{I}{I+1} \left\{ (g_s - g_l) \frac{1}{2} \sum_l (a_{l\,\Omega-1/2}^2 - a_{l\,\Omega+1/2}^2) + g_l\, I + g_R \right\}. \quad (24\,\mathrm{a})$$

It may be pointed out that (24) can be rewritten in the formally simple form

$$\mu = \frac{\Omega K}{I+1} (g_\Omega - g_R) + g_R I, \quad (24\,\mathrm{b})$$

where

$$g_\Omega = \frac{1}{\Omega} \{ g_s \langle s_{z'} \rangle + g_l \langle l_{z'} \rangle \}. \quad (24\,\mathrm{c})$$

For the case $\Omega = K = \frac{1}{2}$ (when some extra terms enter due to the symmetrization of the wave function) equation (24) is somewhat modified

$$= \frac{1}{I+1}\left\{(g_s - g_l)\left[\frac{1}{4}\sum_l (a_{l0}^2 - a_{l1}^2) + (-)^{I-1/2+l}\frac{1}{2}\left(I+\frac{1}{2}\right)\sum_l a_{l0}^2\right]\right.$$
$$\left. + (g_l - g_R)\left[\frac{1}{4} + (-)^{I-1/2}\frac{1}{2}\left(I+\frac{1}{2}\right)a\right] + g_R I(I+1)\right\}, \qquad (25)$$

where a is given by (19). For free nucleons one has $g_s = \begin{pmatrix} 5{,}585 \\ -3{,}826 \end{pmatrix}$ and $g_l = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for protons and neutrons, respectively. Under the assumption that the rotational motion can be described in terms of an irrotational flow of uniformly charged nuclear matter, $g_R \simeq \frac{Z}{A}$.

For the case $I = \Omega = K = \frac{1}{2}$ and odd parity, when $\langle \bar{s}\cdot\bar{I}\rangle$ according to (23b) is independent of the internal wave function, (25) simplifies to

$$\mu = \frac{1}{3}\left\{(g_l - g_R)\,a - \frac{1}{2}g_s + g_l + g_R\right\}, \qquad (26)$$

which means that, for this particular case, there exists between the quantities a and μ a relation that involves only the gyromagnetic ratios g_s, g_l, and g_R, but not the nucleonic wave functions.

c. *Determination of electromagnetic transition probabilities.*

The electric and magnetic multipole operators in the space fixed system (x'', y'', z'') are given by*

$$\mathfrak{M}_e''(\lambda,\mu) = \sum_p \left[e_p + (-)^\lambda \frac{Ze}{A^\lambda}\right]\cdot r_p^\lambda Y_{\lambda\mu}(\vartheta_p'', \varphi_p'') + \frac{3}{4\pi} Z e R_0^\lambda \alpha_{\lambda\mu}^\dagger \qquad (27\,\mathrm{a})$$

$$\mathfrak{M}_m''(\lambda,\mu) = \frac{e\hbar}{2Mc}\sum_p \left(g_s\bar{s} + \frac{2}{\lambda+1}g_l\bar{l}\right)_p \cdot \bar{\nabla}_p\left[r_p^\lambda Y_{\lambda\mu}(\vartheta_p'', \varphi_p'')\right]$$
$$+ \frac{e\hbar}{Mc}\frac{1}{\lambda+1} g_R \int \bar{\mathfrak{R}}(\bar{r})\cdot\bar{\nabla}\left[r^\lambda Y_{\lambda\mu}(\vartheta,\varphi)\right] d\tau. \qquad (27\,\mathrm{b})$$

The first terms in the expressions represent the transition moments of the most loosely bound particles which can be

* BM (VII. 5, 6).

individually excited (thus the sum over p is to be taken only over the transforming nucleons), while the last terms represent the multipole moments generated by the collective motion of the nucleus. The recoil effect of the nuclear core (important for dipole transitions) is included in the particle part of (27 a).

The term $a^{\dagger}_{\lambda\mu}$ in (27 a) is the Hermitian conjugate of the coordinate describing the deformation of the nuclear surface in the coordinate system fixed in space.* R_0 is the nuclear radius. As regards the collective part of the magnetic multipole operator, $\overline{\mathfrak{R}}(\bar{r})$ is the collective angular momentum density, and one has $\int \overline{\mathfrak{R}}(\bar{r})\, d\tau = \overline{R}$. This part is in general difficult to handle, except in the case $\lambda = 1$. In that case, it can be incorporated into the first term simply by changing g_s to $g_s - g_R$ and g_l to $g_l - g_R$.

For the strongly deformed nuclei one can distinguish between particle transitions which are associated with a change in the intrinsic wave function χ_Ω, and collective transitions which leave the internal particle structure unaffected. Of the collective transitions those that have been most studied are the rotational ones which leave φ_{vib} unaffected and only change the rotational state $\mathfrak{D}$ of the system.

The intrinsic structure χ_Ω affects the transition probabilities for particle transitions and for rotational transitions of M1 type. We shall in the following limit ourselves to those cases. We can then simply leave the last term of (27 a) out of consideration since it is effective only in collective transitions (and we shall not consider rotational E2 transitions)**.

It is useful to introduce the reduced transition probability

$$B(\lambda, I \to I') = \sum_{\mu M'} \left| \langle \Omega', I'K'M' \,|\, \mathfrak{M}''(\lambda, \mu) \,|\, \Omega, IKM \rangle \right|^2. \qquad (28)$$

The probability for a γ-transition with a frequency ω, where $\hbar\omega$ is the energy difference between the initial and final state, is then***

* AB (1).

** Cf., however, G. Alaga, K. Alder, A. Bohr, and B. Mottelson (Dan. Mat. Fys. Medd. **29**, no. 9, 1955). In this paper, the authors take into account a small decoupling of the rotational from the intrinsic motion, leaving Ω only approximately a constant of the motion. This effect may render the collective term of (27 a) important for certain particle transitions, particularly of E2 type.

*** BM (VII.1), cf. also J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics, J. Wiley and Sons, New York (1952), chapter XII.

$$T(\lambda) = \frac{8\pi(\lambda+1)}{\lambda[(2\lambda+1)!!]^2}\frac{1}{\hbar}\left(\frac{\omega}{c}\right)^{2\lambda+1} B(\lambda). \tag{29}$$

$B(E\lambda)$ also enters the expression for the $E\lambda$ Coulomb excitation cross sections*.

As pointed out, the coordinates x'' etc. in (27a, b) refer to a coordinate system fixed in space. It is convenient to transform the multipole operators to the coordinate system fixed in the nucleus

$$\mathfrak{M}''(\lambda, \mu) = \sum_{\nu} \mathfrak{D}^{\lambda}_{\mu\nu}(\theta_i)\, \mathfrak{M}'(\lambda, \nu), \tag{30}$$

where $\mathfrak{M}'$ is of the same functional form as $\mathfrak{M}''$ but depends on the new coordinates x'. The functions $\mathfrak{D}$, depending on the Eulerian angles θ_i, are the same as those used in (1).

In the matrix elements in (28) the integration over the Eulerian angles can now be performed and the summation over μ and M' can be carried out.** One then obtains

$$\left.\begin{aligned} B(\lambda, I\rightarrow I') = \Big| &\langle I\lambda K\, K'-K \,|\, I\lambda I'K'\rangle \int \chi'^{\dagger}_{\Omega'}\, \mathfrak{M}(\lambda, K'-K)\, \chi_{\Omega}\, d\tau \\ + &\langle I\lambda K\, -K'-K \,|\, I\lambda I'\, -K'\rangle \int [(-)^{I'-j'} \chi'_{-\Omega'}]^{\dagger}\, \mathfrak{M}(\lambda, -K'-K)\, \chi_{\Omega}\, d\tau \Big|^2. \end{aligned}\right\} \tag{31}$$

The second term contributes only for the empirically rather unusual case $\lambda \geq K + K'$. In evaluating $B(\lambda)$, we have used

$$\int \mathfrak{D}^{I'\dagger}_{M'K'}\, \mathfrak{D}^{\lambda}_{\mu\nu}\, \mathfrak{D}^{I}_{MK}\, d\Omega^3 = \frac{8\pi^2}{2I'+1} \langle I\lambda M\mu \,|\, I\lambda I'M'\rangle \langle I\lambda K\nu \,|\, I\lambda I'K'\rangle, \tag{32}$$

where $d\Omega^3$ signifies integration over all three Eulerian angles. We shall later need the closely related formula

$$\int Y^{\dagger}_{l'\Lambda'}\, Y_{\lambda\nu}\, Y_{l\Lambda}\, d\Omega^2 = \sqrt{\frac{(2l+1)(2\lambda+1)}{4\pi(2l'+1)}} \langle l\lambda\Lambda\nu \,|\, l\lambda l'\Lambda'\rangle \langle l\lambda 0 0 \,|\, l\lambda l' 0\rangle \tag{33}$$

Equation (10) is a special case of (33).

It is of advantage to make the transformation to dimensionless

* BM (Ap VI.17, 18).
** Cf. G. Alaga et al., loc. cit.

variables (6). This gives a factor $\left(\frac{\hbar}{M\omega_0}\right)^{\lambda/2}$ for the electric multipole operator and a factor $\left(\frac{\hbar}{M\omega_0}\right)^{\frac{\lambda-1}{2}}$ for the magnetic multipole operator when r' is replaced by r.

The operator $\bar{l}\cdot(\bar{\nabla}\, r^{\lambda} Y_{\lambda\nu})$ can be rewritten*

$$\left.\begin{aligned} \bar{l}\cdot(\bar{\nabla}\, r^{\lambda} Y_{\lambda\nu}) = \sqrt{\frac{2\lambda+1}{2\lambda-1}}\Big[\sqrt{\lambda^2-\nu^2}\cdot l_z r^{\lambda-1} Y_{\lambda-1\,\nu} + \frac{1}{2}\sqrt{(\lambda-\nu)(\lambda-\nu-1)} \\ \cdot l_- r^{\lambda-1} Y_{\lambda-1\,\nu+1} - \frac{1}{2}\sqrt{(\lambda+\nu)(\lambda+\nu-1)}\, l_+ r^{\lambda-1} Y_{\lambda-1\,\nu-1}\Big] \end{aligned}\right\} \quad (34)$$

where

$$l_+ = l_x + i l_y, \quad l_- = l_x - i l_y. \qquad (34\,\mathrm{a})$$

The same formula holds for $\bar{s}\cdot(\bar{\nabla}\, r^{\lambda} Y_{\lambda\nu})$ with s_- exchanged for l_- etc.

For particle transitions due to electric multipoles, we can write

$$\left.\begin{aligned} B(E\lambda,\ I \to I') = e^2\left(1+(-)^{\lambda}\frac{Z}{A^{\lambda}}\right)^2\left(\frac{\hbar}{M\omega_0}\right)^{\lambda}\frac{2\lambda+1}{4\pi} \\ \cdot\left|\langle I\lambda K K'-K \mid I\lambda I' K'\rangle + b_{E\lambda}(-)^{I'+K'}\langle I\lambda K\, -K'-K \mid I\lambda I'\, -K'\rangle\right|^2\cdot G_{E\lambda}^2 \end{aligned}\right\} \quad (35)$$

where

$$\left.\begin{aligned} b_{E\lambda} = \frac{(-)^{K'+1/2+l'}}{G_{E\lambda}}\Big\{\sum_{l'l}\langle N'l' \mid r^{\lambda} \mid Nl\rangle\sqrt{\frac{2l+1}{2l'+1}}\langle l\lambda 0 0 \mid l\lambda l' 0\rangle \\ \cdot\sum_{\Lambda'\Lambda\Sigma'\Sigma}\delta_{-\Sigma',\Sigma}\, a'_{l'\Lambda'}\, a_{l\Lambda}\langle l\lambda\Lambda\, -K'-K \mid l\lambda l'\, -\Lambda'\rangle\Big\} \end{aligned}\right\} \quad (35\,\mathrm{a})$$

$$\left.\begin{aligned} G_{E\lambda} = \sum_{l'l}\langle N'l' \mid r^{\lambda} \mid Nl\rangle\sqrt{\frac{2l+1}{2l'+1}}\langle l\lambda 0 0 \mid l\lambda l' 0\rangle \\ \cdot\sum_{\Lambda'\Lambda\Sigma'\Sigma}\delta_{\Sigma'\Sigma}\, a'_{l'\Lambda'}\, a_{l\Lambda}\langle l\lambda\Lambda K'-K \mid l\lambda l'\Lambda'\rangle \end{aligned}\right\} \quad (35\,\mathrm{b})$$

* Cf. H. Bethe, Quantenmechanik der Ein- und Zwei-Elektronenprobleme, p. 559, Handbuch der Physik, XXIV/1, Berlin (1933). Note, however, our different choice of phases which agrees with that of E. U. Condon and G. H. Shortley, The Theory of Atomic Spectra. Cf. also M. E. Rose and R. K. Osborn, Phys. Rev., **93**, 1322 (1954).

For transitions due to magnetic multipoles we shall for $\lambda > 1$ omit the last term in (27b), which is expected to have a relatively minor influence. One then obtains

$$\left.\begin{aligned}(M\lambda,\ I\to I') = \left(\frac{e\hbar}{2\,Mc}\right)^2\left(\frac{\hbar}{M\omega_0}\right)^{\lambda-1}\cdot\frac{1}{4}\cdot\frac{2\,\lambda+1}{4\,\pi}\,\big|\langle I\lambda K\ K'-K\,|\,I\lambda I'K'\rangle \\ + b_{M\lambda}(-)^{I'+K'}\langle I\lambda K\ -K'-K\,|\,I\lambda I'\ -K'\rangle\big|^2\,G^2_{M\lambda},\end{aligned}\right\}\quad(36)$$

where

$$\left.\begin{aligned}b_{M\lambda} &= \frac{(-)^{K'+1/2+l'}}{G_{M\lambda}}\sum_{l'l}\langle N'l'|r^{\lambda-1}|Nl\rangle\langle l\,\lambda-1\,0\,0\,|\,l\,\lambda-1\,l'\,0\rangle\sqrt{\frac{2\,l+1}{2\,l'+1}}\\
&\cdot\sum_{\Lambda'\Lambda\Sigma'\Sigma}a'_{l'\Lambda'}a_{l\Lambda}\cdot\Big\{g_s\Big[A(q)\,\delta_{-\Sigma',\Sigma}(-)^{\Sigma-\frac{1}{2}}\langle l\,\lambda-1\,\Lambda\,q\,|\,l\,\lambda-1\,l'\ -\Lambda'\rangle\\
&\quad+B(q)\,\delta_{\Sigma',\frac{1}{2}}\,\delta_{\Sigma,\frac{1}{2}}\langle l\,\lambda-1\,\Lambda\,q+1\,|\,l\,\lambda-1\,l'\ -\Lambda'\rangle\\
&\quad-C(q)\,\delta_{\Sigma',-\frac{1}{2}}\,\delta_{\Sigma,-\frac{1}{2}}\langle l\,\lambda-1\,\Lambda\,q-1\,|\,l\,\lambda-1\,l'\ -\Lambda'\rangle\Big]\\
&\quad+\frac{2}{\lambda+1}\,g_l\,\delta_{-\Sigma',\Sigma}\Big[A(q)\cdot(-2\,\Lambda')\langle l\,\lambda-1\,\Lambda\,q\,|\,l\,\lambda-1\,l'\ -\Lambda'\rangle\\
&\quad+B(q)\sqrt{(l'+\Lambda')(l'-\Lambda'+1)}\langle l\,\lambda-1\,\Lambda\,q+1\,|\,l\,\lambda-1\,l'\ -\Lambda'+1\rangle\\
&\quad-C(q)\sqrt{(l'-\Lambda')(l'+\Lambda'+1)}\langle l\,\lambda-1\,\Lambda\,q-1\,|\,l\,\lambda-1\,l'\ -\Lambda'-1\rangle\Big]\Big\}\end{aligned}\right\}\quad(36\,\mathrm{a})$$

$$\left.\begin{aligned}G_{M\lambda} &= \sum_{l'l}\langle N'l'|r^{\lambda-1}|Nl\rangle\langle l\lambda-1\,0\,0\,|\,l\,\lambda-1\,l'\,0\rangle\\
&\cdot\sqrt{\frac{2\,l+1}{2\,l'+1}}\sum_{\Lambda'\Lambda\Sigma'\Sigma}a'_{l'\Lambda'}a_{l\Lambda}\cdot\Big\{g_s\Big[A(k)\,\delta_{\Sigma',\Sigma}(-)^{\Sigma-\frac{1}{2}}\langle l\,\lambda-1\,\Lambda\,k\,|\,l\,\lambda-1\,l'\Lambda'\rangle\\
&\quad+B(k)\,\delta_{\Sigma',-\frac{1}{2}}\,\delta_{\Sigma,\frac{1}{2}}\langle l\,\lambda-1\,\Lambda\,k+1\,|\,l\,\lambda-1\,l'\Lambda'\rangle\\
&\quad-C(k)\,\delta_{\Sigma',\frac{1}{2}}\,\delta_{\Sigma,-\frac{1}{2}}\langle l\,\lambda-1\,\Lambda\,k-1\,|\,l\,\lambda-1\,l'\Lambda'\rangle\Big]\\
&\quad+\frac{2}{\lambda+1}\,g_l\,\delta_{\Sigma',\Sigma}\Big[A(k)\cdot 2\,\Lambda'\langle l\,\lambda-1\,\Lambda\,k\,|\,l\,\lambda-1\,l'\Lambda'\rangle\\
&\quad+B(k)\sqrt{(l'-\Lambda')(l'+\Lambda'+1)}\langle l\,\lambda-1\,\Lambda\,k+1\,|\,l\,\lambda-1\,l'\,\Lambda'+1\rangle\\
&\quad-C(k)\sqrt{(l'+\Lambda')(l'-\Lambda'+1)}\langle l\,\lambda-1\,\Lambda\,k-1\,|\,l\,\lambda-1\,l'\Lambda'-1\rangle\Big]\Big\}\end{aligned}\right\}\quad(36\,\mathrm{b})$$

and where in turn

$$A(\nu) = \sqrt{\lambda^2 - \nu^2} \tag{36c}$$

$$B(\nu) = \sqrt{(\lambda - \nu)(\lambda - \nu - 1)} \tag{36d}$$

$$C(\nu) = \sqrt{(\lambda + \nu)(\lambda + \nu - 1)} \tag{36e}$$

$$k = K' - K \tag{36f}$$

$$q = -K' - K. \tag{36g}$$

For $\lambda = 1$, as pointed out above, we can easily handle the last term of (27b) and incorporate it in the first term. All the expressions derived for $\lambda > 1$ are then valid if g_s is replaced by $(g_s - g_R)$ and g_l by $(g_l - g_R)$ everywhere.

For M1 transitions within one rotational band, equations (36) simplify greatly, and one obtains

$$G_{M1} = (g_\Omega - g_R) \cdot 2\,\Omega. \tag{37a}$$

Further, b_{M1} is different from zero only if $\Omega = K = \frac{1}{2}$. (For this latter case we denote b_{M1} by $b_0\sqrt{2}$ and G_{M1} by G_0.) One has

$$b_0 = -\frac{(-)^l}{g_\Omega - g_R}\left\{(g_s - g_R)\sum_l a_{l0}^2 + 2\,(g_l - g_R)\sum_l \sqrt{l(l+1)}\,a_{l0}\,a_{l1}\right\}. \tag{37b}$$

The reduced transition probability for a transition of this kind from a level $I' + 1$ to a level I' (both belonging to the same rotational band) has then the simple form

$$B_0(\mathrm{M}\,1) = \frac{3}{64\,\pi}\left(\frac{e\hbar}{2\,Mc}\right)^2 \frac{2\,I' + 1}{I' + 1}\,G_0^2 \cdot \left|\,1 + b_0\,(-)^{I' - 1/2}\right|^2. \tag{37c}$$

For a rotational band with $\Omega = \frac{1}{2}$, we have given expressions for the four measurable quantities a, μ, G_0, and b_0 in (19), (25), (37a), and (37b), respectively. As the dependence of the internal wave function on all these quantities is contained in the expressions $\sum_l a_{l0}^2$ and $\sum_l \sqrt{l(l+1)}\,a_{l0}\,a_{l1}$, it is apparent that between a, μ, G_0, and b_0 there must exist relations that are independent of the nucleonic structure.

We have already found such a relation between a and μ for $I = \Omega = K = \frac{1}{2}$ and odd parity, in which case formula (26) holds. One can further for this case derive the second relation

$$2\, b_0 G_0 = G_0 - 2\,(g_l - g_R)\, a + g_s - 2\, g_l + g_R. \tag{38}$$

For the case $I = \Omega = K = \frac{1}{2}$ and even parity the two corresponding relations are

$$G_0 = 3\,\mu - a\,(g_l - g_R) - \frac{1}{2} g_s + g_l - 2\, g_R \tag{39}$$

and

$$b_0 = -\frac{1}{2\, G_0}\left[3\,\mu + a\,(g_l - g_R) + \frac{1}{2} g_s - g_l - g_R\right]. \tag{40}$$

For the case $I \neq \Omega = K = \frac{1}{2}$, one can also establish relations of the same kind as (39) and (40).

The radial matrix element $\langle N'l' \,|\, r^{\lambda} \,|\, Nl \rangle$ is given by the formula*

$$\left.\begin{aligned} \langle N'l' \,|\, r^{\lambda} \,|\, Nl \rangle &= \left[\frac{\Gamma(n+1)\,\Gamma(n'+1)}{\Gamma(n+t-\nu+1)\cdot\Gamma(n'+t-\nu'+1)}\right]^{1/2} \nu'!\,\nu!\\ &\sum_{\sigma}{}' \frac{\Gamma(t+\sigma+1)}{\sigma!\,(n-\sigma)!\,(n'-\sigma)!\,(\sigma+\nu-n)!\,(\sigma+\nu'-n')!}, \end{aligned}\right\} \tag{41}$$

where

$$n = \frac{1}{2}(N - l) \tag{41 a}$$

$$n' = \frac{1}{2}(N' - l') \tag{41 b}$$

$$\nu = \frac{1}{2}(l' - l + \lambda) \tag{41 c}$$

$$\nu' = \frac{1}{2}(l - l' + \lambda) \tag{41 d}$$

* See, e. g., P. Morse and H. Feshbach, Methods of Theoretical Physics, McGraw-Hill (1953), p. 785. The extra phase factor appearing in this reference is due to the fact that $|\,Nl\,\rangle_{MF} \sim (-)^n\,|\,Nl\,\rangle_{\text{here}}$. Formulae (11 a–e) are given from (41) for $\lambda = 2$.

$$t = \frac{1}{2}(l + l' + \lambda + 1), \tag{41e}$$

and where the condition on the summation variable σ is

$$\begin{matrix} n \\ n' \end{matrix} \geqslant \sigma \geqslant \begin{matrix} n - \nu \\ n' - \nu' . \end{matrix} \tag{42}$$

This means that σ has to be smaller than or equal to the smallest of n and n' etc. If this condition cannot be fulfilled by any σ, the integral vanishes. An equivalent necessary condition (expressed in N, l, and λ) for the matrix element of r^{λ} to be different from zero can be formulated as

$$l + \lambda \geqslant l' \geqslant l - \lambda \tag{42a}$$

$$N + \lambda \geqslant N' \geqslant N - \lambda . \tag{42b}$$

e. *Determination of ft-values for beta transitions.*

As it is the purpose of this paragraph merely to illustrate the application of the strong coupling wave functions in the field of beta transitions, we limit ourselves to considering only allowed transitions and a select group of forbidden transitions, namely those which imply a parity change of $(-)^{\Delta I + 1}$ (with $I \neq 0$). This latter group is of a pure Gamow-Teller type.

The treatment of β-transitions is similar to that of γ-transitions and it is useful to introduce the concept of reduced transition probability* defined in analogy to (28)

$$D_{\substack{F\\GT}}(n) = \sum_{\mu M'} \left| \langle \Omega', \; I'M'K' \,|\, \mathfrak{D}_{\substack{F\\GT}}(n, \mu) \,|\, \Omega, IMK \rangle \right|^2. \tag{43}$$

Here $\mathfrak{D}_{\substack{F\\GT}}(n, \mu)$ is the Fermi respectively the Gamow-Teller transition operator, and n is the degree of forbiddenness, $n = \Delta I - 1$.

The comparative half lives or the ft-values can now be defined in terms of these reduced transition probabilities.

* See BM, chapter VIII.

For allowed transitions we can write

$$f_0 t = B_g \left[(1 - x)\, D_F(0) + x\, D_{GT}(0)\right]^{-1}, \tag{44}$$

where t is the half life, f_0 the integrated Fermi function for allowed transitions, $g\,(1-x)^{1/2}$ and $g x^{1/2}$ are the Fermi and Gamow-Teller coupling constants, and the constant B_g is given as

$$B_g = \frac{2\,\pi^3 \hbar^7 \ln 2}{g^2 m_e^5 e^4}. \tag{45}$$

For forbidden transitions of the particular type considered here (parity change $= (-)^{\triangle I + 1}$ etc.) one has

$$f_n t = B_g \left[x D_{GT}(n)\right]^{-1}, \tag{46}$$

where f_n is the integrated Fermi function corresponding to the order of forbiddenness n [for definition and normalization see BM (VIII.6)].

It turns out that $D_{GT}(n)$ has a structure very similar to the reduced transition probability for a γ-transition of the magnetic multipole type with $\lambda = n + 1$. The corresponding operator is defined as

$$\mathfrak{D}_{GT}(n, \mu) = S(n) \sum_p \bar{s}_p \cdot \overline{\nabla}_p \left[r_p^{n+1} Y_{n+1\,\mu}(\vartheta_p, \varphi_p)\right] \tau_\pm^p, \tag{47}$$

where

$$S(n) = \left[\frac{4\,\pi\, 2^{n+3}}{(2\,n+3)!}\right]^{1/2} \frac{[(n+1)!]^2}{n+1} \left(\frac{mc}{\hbar}\right)^n. \tag{47 a}$$

In general the sum over p is to be taken over all particles involved in the transition. We restrict ourselves, however, to transitions between odd-A nuclei with only one unpaired particle which then undergoes the β-transition. τ_+ and τ_- are the isotopic spin creation and annihilation operators, transforming a proton into a neutron, and vice versa.

By formally putting $g_s = 1$ and $g_l = g_R = 0$ in (27 b) we obtain exactly the above expression apart from a multiplicative numerical factor and the isotopic spin operator. Making this formal change we can then use formula (36) of the preceding paragraph for calculating $D_{GT}(n)$. Thus,

$$
\left.\begin{aligned}
D_{GT}(n) &= S(n)^2\left(\frac{\hbar}{M\omega_0}\right)^n \frac{1}{4}\,\frac{2n+3}{4\pi} \\
&\cdot \Big| \langle I\, n+1\, K\, K'-K \,|\, I\, n+1\, I' K' \rangle \\
+ \beta_{n+1}(-)^{I'+K'} &\langle I\, n+1\, K\, -K'-K \,|\, I\, n+1\, I'\, -K' \rangle \Big|^2 \gamma_{n+1}^2
\end{aligned}\right\} \quad (48\,\mathrm{a})
$$

where

$$
\gamma_{n+1} = G_{M\,n+1}(g_s = 1,\, g_l = g_R = 0) \quad (48\,\mathrm{b})
$$

$$
\beta_{n+1} = b_{M\,n+1}(g_s = 1,\, g_l = g_R = 0). \quad (48\,\mathrm{c})
$$

For the case of mirror transitions, this simplifies to

$$
D_{GT}(0) = \Big| \langle I\, 1\, K\, 0 \,|\, I\, 1\, I\, K \rangle + \beta_1 (-)^{I+K} \langle I\, 1\, K\, -1 \,|\, I\, 1\, I\, -K \rangle \Big|^2 \gamma_1^2, \quad (48\,\mathrm{d})
$$

where

$$
\gamma_1 = \sum_l \left(a_{l\,\Omega-1/2}^2 - a_{l\,\Omega+1/2}^2\right) = 2\langle s_{z'} \rangle \quad (48\,\mathrm{e})
$$

$$
\beta_1 = \frac{(-)^{l+1}}{\gamma_1}\sqrt{2}\sum_l a_{l0}^2\, \delta_{\Omega,\,1/2}\, \delta_{K,\,1/2}. \quad (48\,\mathrm{f})
$$

Finally, in this same case, the Fermi part $D_F(0)$ is simply

$$
D_F(0) = \sum_{M'} \left| \langle \Omega,\, IM'K \,|\, \tau_{\pm}^{p} \,|\, \Omega,\, IMK \rangle \right|^2 = 1. \quad (49)
$$

Collecting the terms, we can write for mirror transitions in odd-A nuclei

$$
f_0 t = B_g \left\{ (1-x)\cdot 1 + x\,\frac{K^2}{I(I+1)}\left[1 + \beta_1 (-)^{I+1/2}\sqrt{2}\left(I+\frac{1}{2}\right)\right]^2 \gamma_1^2 \right\}^{-1}, \quad (50)
$$

where B_g is given by (45).

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CERN (European Organization for Nuclear Research)
Theoretical Study Division at the Institute for
Theoretical Physics, University of Copenhagen,
and
The Institute of Theoretical Physics,
University of Lund.

V. Appendix A.

Use of an Alternative Representation.

In the diagonalization of the Hamiltonian (2), cross terms with different total quantum number N have been neglected. It is possible, however, to obtain an improved solution by making a small change of representation and in this manner to exhibit in a simple way the effect of the neglected non-diagonal terms in N.

Starting from (2) and (2a)

$$H = H_0 + C\bar{l}\cdot\bar{s} + D\bar{l}^2 \qquad \text{(A 1)}$$

$$H_0 = -\frac{\hbar^2}{2M}\Delta' + \frac{M}{2}(\omega_x^2 x'^2 + \omega_y^2 y'^2 + \omega_z^2 z'^2), \qquad \text{(A 1a)}$$

we make a slight parameter change and define ε and $\omega_0(\varepsilon)$, differently from δ and $\omega_0(\delta)$, as

$$\omega_x = \omega_y = \omega_0(\varepsilon)\left(1 + \frac{1}{3}\varepsilon\right) \qquad \text{(A 2a)}$$

$$\omega_z = \omega_0(\varepsilon)\left(1 - \frac{2}{3}\varepsilon\right). \qquad \text{(A 2b)}$$

The following relations hold between the old and the new parameters

$$\varepsilon = \delta + \frac{1}{6}\delta^2 + O(\delta^3) \tag{A 3}$$

$$\omega_0(\varepsilon) = \omega_0(\delta(\varepsilon))\left[1 - \frac{1}{9}\varepsilon^2 + O(\varepsilon^3)\right] = \mathring{\omega}_0\left[1 + \frac{1}{9}\varepsilon^2 + O(\varepsilon^3)\right]. \tag{A 4}$$

We further perform a coordinate transformation

$$\xi = x'\sqrt{\frac{M\omega_x}{\hbar}} \tag{A 5a}$$

$$\eta = y'\sqrt{\frac{M\omega_y}{\hbar}} \tag{A 5b}$$

$$\zeta = z'\sqrt{\frac{M\omega_z}{\hbar}} \tag{A 5c}$$

H_0 is then separable in ξ, η, ζ

$$H_0 = H_\xi + H_\eta + H_\zeta, \tag{A 6}$$

where

$$H_\xi = \frac{1}{2}\hbar\omega_x\left(-\frac{\partial^2}{\partial\xi^2} + \xi^2\right) \text{ etc.} \tag{A 6a}$$

For later use a note should be made at this point that a representation that obviously makes H_0 diagonal is $|n_\xi\rangle|n_\eta\rangle|n_\zeta\rangle$, where $|n_\xi\rangle$ is defined by

$$H_\xi|n_\xi\rangle = \left(n_\xi + \frac{1}{2}\right)\hbar\omega_x|n_\xi\rangle. \tag{A 7}$$

We proceed, however, to split H_0 in a manner analogous to (7)

$$H_0 = \mathring{H}_0 + H_\varepsilon, \tag{A 8}$$

where

$$\mathring{H}_0 = \hbar\omega_0(\varepsilon)\frac{1}{2}[-\Delta_\xi + \varrho^2] \tag{A 8a}$$

$$H_\varepsilon = \frac{1}{6}\varepsilon\hbar\omega_0(\varepsilon)\left[\left(-\frac{\partial^2}{\partial\xi^2} + \xi^2\right) + \left(-\frac{\partial^2}{\partial\eta^2} + \eta^2\right) - 2\left(-\frac{\partial^2}{\partial\zeta^2} + \zeta^2\right)\right], \tag{A 8b}$$

and

$$\varrho^2 = \xi^2 + \eta^2 + \zeta^2.$$

In conformity with the new coordinates ξ, η, ζ introduced it is useful also to introduce an operator $\bar{l}_t$ defined analogously to $\bar{l}$ as

$$(l_t)_x = -i\left(\eta \frac{\partial}{\partial \zeta} - \zeta \frac{\partial}{\partial \eta}\right) \text{ etc.} \tag{A 9}$$

(We denote the component by $(l_t)_x$ instead of $(l_t)_\xi$ to emphasize that the directions of the new coordinates coincide with the old ones.)

We now introduce a representation which makes $\mathring{H}_0$ diagonal together with $(l_t)_z$ and $\bar{l}_t^2$, s_z and $\bar{s}^2$. The eigenvalues of $(l_t)_z$ and $\bar{l}_t^2$ are denoted as Λ_t and $l_t\,(l_t + 1)$.

Thus,

$$\left.\begin{aligned} \mathring{H}_0 \,|\, N_t l_t \Lambda_t \Sigma\rangle &= \hbar\omega_0 \frac{1}{2}(-\Delta_\xi + \varrho^2)\,|\, N_t l_t \Lambda_t \Sigma\rangle \\ &= \left(N_t + \frac{3}{2}\right)\hbar\omega_0 \,|\, N_t l_t \Lambda_t \Sigma\rangle. \end{aligned}\right\} \tag{A 10}$$

We rewrite (A 1) in the form

$$H = H_t + H_{\text{pert}}, \tag{A 11}$$

where

$$H_t = \mathring{H}_0 + H_\varepsilon + C\bar{l}_t\cdot\bar{s} + D\bar{l}_t^2 \tag{A 11a}$$

and

$$H_{\text{pert}} = C\,(\bar{l} - \bar{l}_t)\cdot\bar{s} + D\,(\bar{l}^2 - \bar{l}_t^2). \tag{A 11b}$$

By using the identity

$$\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - 2\frac{\partial^2}{\partial\zeta^2} = \frac{1}{8}[\Delta, [\Delta, \xi^2 + \eta^2 - 2\zeta^2]]$$

and exploiting (A 10), one can show that

$$\left.\begin{aligned} &\langle N_t' l_t' \Lambda_t' \Sigma' \,|\, H_\varepsilon \,|\, N_t l_t \Lambda_t \Sigma\rangle \\ &= \delta_{N_t N_t'} \langle N_t l_t' \Lambda_t' \Sigma' \,|\, \frac{1}{3}\varepsilon\hbar\omega_0(\xi^2 + \eta^2 - 2\zeta^2)\,|\, N_t l_t \Lambda_t \Sigma\rangle. \end{aligned}\right\} \tag{A 12}$$

The fact that H_ε has vanishing matrix elements between states of different N_t can also be seen from (A 8b), (A 6a), and (A 7), remembering that the $| N_t l_t \Lambda_t \rangle$-vector is a particular sum of $| n_\xi \rangle | n_\eta \rangle | n_\zeta \rangle$ product vectors with $n_\xi + n_\eta + n_\zeta = N_t$.

It follows now that H_t has the same matrix elements in the $| N_t l_t \Lambda_t \Sigma \rangle$-representation as H has in the $| Nl\Lambda\Sigma \rangle$-representation apart from, first, the change of parameters (ε and $\omega_0(\varepsilon)$ in the former representation and δ and $\omega_0(\delta)$ in the latter one) and, secondly, the fact that the matrix elements of H_t between states with N_t differing by two vanish identically in the $| N_t l_t \Lambda_t \Sigma \rangle$-representation.

The next step is to investigate the effect of the H_{pert}-term. The three $\bar{l}$-components $l_+ (= l_x + il_y)$, $l_- (= l_x - il_y)$, and l_z may be transformed as follows

$$l_+ = a(l_t)_+ - bf_+ \tag{A 13a}$$

$$l_- = a(l_t)_- - bf_- \tag{A 13b}$$

$$l_z = (l_t)_z, \tag{A 13c}$$

where

$$a = \frac{1}{2}\left[\sqrt{\frac{1+\frac{1}{3}\varepsilon}{1-\frac{2}{3}\varepsilon}} + \sqrt{\frac{1-\frac{2}{3}\varepsilon}{1+\frac{1}{3}\varepsilon}}\right] = 1 + \frac{1}{8}\varepsilon^2 + O(\varepsilon^3) \tag{A 13d}$$

$$b = \frac{1}{2}\left[\sqrt{\frac{1+\frac{1}{3}\varepsilon}{1-\frac{2}{3}\varepsilon}} - \sqrt{\frac{1-\frac{2}{3}\varepsilon}{1+\frac{1}{3}\varepsilon}}\right] = \frac{1}{2}\varepsilon + \frac{1}{12}\varepsilon^2 + O(\varepsilon^3), \tag{A 13e}$$

and where the operators f_+ and f_- can be conveniently written in the form

$$f_+ = \frac{1}{2}[\Delta, -\zeta(\xi + i\eta)] = \sqrt{\frac{2\pi}{15}}[\Delta, \varrho^2 U_{21}] \tag{A 14a}$$

$$f_- = \frac{1}{2}[\Delta, +\zeta(\xi - i\eta)] = \sqrt{\frac{2\pi}{15}}[\Delta, \varrho^2 U_{2-1}]. \tag{A 14b}$$

Here U_{21} is the normalized spherical harmonic of order 2,1, expressed in the angles of the coordinate system ξ, η, ζ.

From (A 10) and (14a, b) one finds

$$\langle N_t \,|\, f_\pm \,|\, N_t \rangle = \sqrt{\frac{8\pi}{15}}\,(N_t' - N_t)\,\langle N_t' \,|\, \varrho^2\, U_{2\,\pm 1} \,|\, N_t \rangle. \quad \text{(A 15)}$$

Next, using (13a, b) and expanding in powers of ε, one can show that

$$H_{\text{pert}} = \varepsilon H_1 + \varepsilon^2 H_2 + \cdots, \quad \text{(A 16)}$$

where

$$H_1 = -\frac{1}{4} C \{ f_+ s_- + f_- s_+ \} - \frac{1}{2} D \{ (l_t)_- f_+ + f_- (l_t)_+ \} \quad \text{(A 16a)}$$

and

$$\left.\begin{aligned} H_2 = \frac{1}{16} C &\left\{ (l_t)_+ s_- + (l_t)_- s_+ - \frac{3}{2}(f_+ s_- + f_- s_+) \right\} \\ + \frac{1}{4} D &\left\{ (l_t)_- (l_t)_+ + f_- f_+ - \frac{1}{3}[(l_t)_- f_+ + f_- (l_t)_+] \right\}. \end{aligned}\right\} \quad \text{(A 16b)}$$

Now it follows from (A 15) that the matrix elements of εH_1 between states of the same N_t vanish. On the other hand, εH_1 causes a coupling in the $|\, N_t l_t \Lambda_t \Sigma \rangle$-representation between states differing by two in their N_t-value of formally a very similar kind [cf. (A 15) and (7 b)] to the coupling caused by H_δ in the $|\, N l \Lambda \Sigma \rangle$-representation between the N and $N+2$ shells. An estimate of the εH_1 coupling terms shows, however, that their order of magnitude is only 1/10 of the H_δ coupling terms, or something similar to the ratio of the matrix elements of the $\bar{l} \cdot \bar{s}$- and $\bar{l}^2$-terms to the matrix elements of H_0.

The second order terms in (A 16) amount only to the order of a per cent of the total $\bar{l} \cdot \bar{s}$- and $\bar{l}^2$-terms and are therefore negligible.

By interpreting the representatives $A_{l\Lambda}$, listed in Table I, as being representatives $A_{l_t \Lambda_t}$ of eigenfunctions in the $|\, N_t l_t \Lambda_t \Sigma \rangle$-representation, one should thus obtain an improved approximation.

The uncorrected more simple eigenfunctions are, however, used in the main part of this paper since they are sufficiently accurate for most of the applications. Thus, the matrix elements for operators as μ and a and those involved in M1 transitions, which all contain exclusively $\bar{l}$ and $\bar{s}$, are affected only to the order ε^2 by the change in the wave functions. Moreover, the correction term of this order involves a small coefficient [cf. (A 13d)] and is therefore of little significance. The situation is somewhat different for operators like the quadrupole moment and E2 transition operators, and in an estimate of matrix elements of these quantities it may sometimes be important to use the improved representation.

Finally, it may be added that, apart from the smallness of H_{pert}, it is even questionable which Hamiltonian H or H_t best describes the nuclear conditions.

The $\bar{l}\cdot\bar{s}$-term for the nuclear case is modelled after the spin-orbit coupling term for an electron moving in an electrostatic field. This term is of the form $\bar{s}\cdot(\bar{v}\times\text{grad } V)$ which only for an isotropic potential reduces to $\bar{l}\cdot\bar{s}$. For a deformed oscillator potential, the difference between such a coupling term and the $\bar{l}\cdot\bar{s}$-term is of the same order of magnitude as the difference between $\bar{l}\cdot\bar{s}$ and $\bar{l}_t\cdot\bar{s}$.

Similarly for the $\bar{l}^2$-term, which is thought of as a correction at larger distances for the too fast rising of the oscillator walls, there is no reason to assume a spherically symmetric correction when the oscillator potential itself becomes eliptically deformed.

Thus, it appears that the effect of H_{pert} lies entirely within the range of ambiguity in the definition of the $\bar{l}\cdot\bar{s}$- and $\bar{l}^2$-terms for the deformed nucleus.

Appendix B.

Asymptotic Solutions in the Limit of Very Strong Deformations.

The notation and parameters used in this section are identical with those employed in Appendix A.

We first consider the Hamiltonian H_0 (A 1a) containing only kinetic energy and oscillator field terms. It follows from (A 6, A 7) that the energy eigenvalues corresponding to H_0 are of the form

$$E_0 = \left(n_\zeta + \frac{1}{2}\right)\hbar\omega_z + (n_\xi + n_\eta + 1)\,\hbar\omega_x,$$

which can be rewritten in terms of the deformation parameter ε [cf. (A 2a, b)]

$$E_0 = \hbar\omega_0(\varepsilon)\left[\left(N + \frac{3}{2}\right) + \varepsilon\,\frac{n_\perp - 2\,n_\zeta}{3}\right], \qquad \text{(B 1)}$$

where $n_\perp = n_\xi + n_\eta$. The energy eigenvalues in units of $\hbar\omega_0(\varepsilon)$, plotted as functions of ε, are then straight lines. The corresponding eigenfunctions are $|\,n_\xi\rangle\,|\,n_\eta\rangle\,|\,n_\zeta\rangle$.

Such a level characterized by n_ζ and $n_\perp$ is degenerate to the order $n_\perp + 1$ (number of combinations of n_ξ and n_η that fulfil $n_\xi + n_\eta = n_\perp$). To this degeneracy is then added the spin degeneracy.

We further introduce linear combinations $|\,n_\perp\Lambda\rangle$ of base vectors $|\,n_\xi\rangle\,|\,n_\eta\rangle$ (with $n_\xi + n_\eta = n_\perp$) such that

$$[(l_t)_z - \Lambda]\,|\,n_\perp\Lambda\rangle = 0.$$

The vectors $|\,n_\zeta\rangle\,|\,n_\perp\Lambda\rangle\,|\,\Sigma\rangle$ form a complete set, and H_0 is further diagonal in such a representation. Here $\Lambda = \pm 1, \pm 3, \ldots, \pm n_\perp$ if $n_\perp$ is odd, and $= 0, \pm 2, \ldots, \pm n_\perp$ if $n_\perp$ is even.

We now consider elements of $\bar{l}_t\cdot\bar{s}$ and $\bar{l}_t^2$ in this representation. As before, we can have coupling terms only between states of the same Ω and N $(= n_\zeta + n_\perp)$. Apart from diagonal elements, non-vanishing matrix elements of $\bar{l}_t\cdot\bar{s}$ occur only between states differing by one unit in Λ and $n_\perp$. As regards $\bar{l}_t^2$, this operator is diagonal in Λ and has non-vanishing elements only between states with $n_\perp$ equal or different by two.

The diagonal elements of $\bar{l}_t\cdot\bar{s}$ are given immediately as

$$\langle\, n_\zeta\, n_\perp\, \Lambda\, \Sigma\,|\,\bar{l}_t\cdot\bar{s}\,|\,n_\zeta\, n_\perp\, \Lambda\, \Sigma\,\rangle = \Lambda\,\Sigma. \qquad \text{(B 2)}$$

Employing operator relations of the type used in Appendix A, one can show

$$\langle\, n_\zeta\, n_\perp\, \Lambda\, \Sigma\,|\,\bar{l}_t^2\,|\,n_\zeta\, n_\perp\, \Lambda\, \Sigma\,\rangle = \Lambda^2 + 2\,n_\perp\,n_\zeta + 2\,n_\zeta + n_\perp. \qquad \text{(B 3)}$$

Figs. 4a and 4b give a comparison of the energy levels of the $N = 5$ shell by perturbation treatment and exact calculation, for the largest deformations calculated in the numerical treatment. The level group (α) corresponds to the harmonic oscillator levels (B1) while (β) employs the diagonal terms of (B2) and (B3). Finally (γ) shows the exact levels.

These asymptotic solutions corresponding to the $| n_\zeta n_\perp \Lambda \Sigma \rangle$-states may be of interest in providing new approximate selection rules for particle transitions in this region of deformation, connected with the occurrence of the new constants of the motion $n_\perp$ and Σ.

Appendix C.

The Total Energy as Function of the Deformation Parameter.

We shall here evaluate the expectation value of the total energy Hamiltonian $\mathfrak{H}$, defined by (15), employing the notation and results of Appendix A, i. e. taking into account the effect on the wave functions of the coupling between shells characterized by different N-values. We write

$$\mathfrak{H} = \frac{1}{2}\sum_i H_i + \frac{1}{2}\sum_i T_i = \frac{3}{4}\sum_i H_i - \frac{1}{4}\sum_i (V_i - T_i).$$

The eigenvalues of H_i are just the calculated single-particle energy eigenvalues E_i. Separating out l-dependent terms of the difference $V_i - T_i$ we can write

$$V_i - T_i = W_i + U_i, \tag{C1}$$

where in the notation of Appendix A (dropping the index i)

$$W = \hbar\omega_x \left[\frac{\partial^2}{\partial\xi^2} + \xi^2 + \frac{\partial^2}{\partial\eta^2} + \eta^2\right] + \hbar\omega_z \left[\frac{\partial^2}{\partial\zeta^2} + \zeta^2\right] \tag{C1a}$$

$$U = C\,\bar{l}\cdot\bar{s} + D\,\bar{l}^2 \simeq C\,\bar{l}_t\cdot\bar{s} + D\,\bar{l}_t^2. \tag{C1b}$$

Noting that the single-particle wave functions can be written as linear combinations of $| n_\xi \rangle \, | n_\eta \rangle \, | n_\zeta \rangle$, with the requirement $n_\xi + n_\eta + n_\zeta = N_t$, it follows immediately from the virial theorem for one-dimensional harmonic oscillators that

$$\langle W_i \rangle = 0. \tag{C2}$$

Figs. 4a and 4b further demonstrate that U_i is approximately diagonal at the largest deformations with respect to the wave functions which are appropriate at these deformations. The approximate expectation values of U_i can be calculed from (B 2) and (B 3). To the extent this approximation is valid, U_i is independent of the deformation.

Using the equivalent of formula (13) (employing ε instead of δ) and formula (A 4), one finally obtains *, **

$$\mathfrak{E}(\varepsilon) = \frac{3}{4}\hbar\mathring{\omega}_0\left\{\sum_i\left(N_i+\frac{3}{2}\right)\left(1+\frac{1}{9}\varepsilon^2\right)+\sum_i \varkappa r_i(\varepsilon)\right\}+\frac{1}{4}\sum_i\langle U_i\rangle. \qquad (C\,3)$$

The equilibrium deformation ε_{eq} is then obtained by solving

$$\frac{\partial\mathfrak{E}(\varepsilon)}{\partial\varepsilon} = 0.$$

The relation between the deformation parameters ε and δ, employed here and in the main text, respectively, is given by (A 3).

* A correction to (C3) is obtained by considering the diagonal terms of the neglected Coulomb interaction between the protons. This effect will tend to increase the equilibrium deformation. For, e. g., a homogeneously charged ellipsoid of an average radius R_0 one has to second order in ε

$$E_{el} = \frac{3}{5}\frac{Z^2e^2}{R_0}\left(1-\frac{4}{45}\varepsilon^2\right).$$

(Cf., e. g., N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).)

The ε^2-dependent term is negligible for the lighter nuclei, and even for nuclei around $A = 200$ it does not amount to more than 10 per cent of the "surface tension" term (second order term) in (C3).

** Note added in proof: It is possible to estimate the effect of the residual interactions by employing the two-nucleon model used by A. Bohr and B. Mottelson (Dan. Mat. Fys. Medd., in press). These interactions tend always to reduce the deformation from that calculated for completely independent particle motion as above. The effect becomes less important for increasing deformation. For $\varepsilon = 0.3$ and a strength of interaction $v = 0.3$, as defined in the above reference, the equilibrium deformation is reduced by 10 per cent. (Private communication from B. Mottelson.)

Tables.

TABLE I. Eigenvalues and Eigenfunctions for the Deformed Field.

$N = 0 \quad \Omega = \frac{1}{2}$

eigenvalue: $r = 0$, eigenvector: $|\,000 + \rangle$.

$N = 1 \quad \Omega = \frac{3}{2}$

eigenvalue: $r = \frac{1}{3}\eta - 1$, eigenvector: $|\,111 + \rangle$.

	$\eta = -6$	-4	-2	2	4	6
2	-3.000	-2.333	-1.667	-0.333	0.333	1.000

$N = 1 \quad \Omega = \frac{1}{2}$

base vectors: $|\,110 + \rangle$, $|\,111 - \rangle$.

	$\eta = -6$	-4	-2	2	4	6
4	4.372 1.000 -0.263	3.228 1.000 -0.397	2.333 1.000 -0.707	2.228 1.000 -2.518	2.706 1.000 -3.798	3.275 1.000 -5.144
3	-1.372 1.000 3.798	-0.895 1.000 2.518	-0.667 1.000 1.414	-1.895 1.000 0.397	-3.039 1.000 0.263	-4.275 1.000 0.194

$N = 2 \quad \Omega = \frac{5}{2}$

eigenvalue: $r = \frac{2}{3}\eta - 2$, eigenvector: $|\,222 + \rangle$.

	$\eta = -6$	-4	-2	2	4	6
5	-6.000	-4.667	-3.333	-0.667	0.667	2.000

$N = 2 \quad \Omega = \frac{3}{2}$

base vectors: $|221+\rangle$, $|222-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
8	2.000	1.895	2.228	4.035	5.198	6.424
	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.500	— 0.781	— 1.281	— 2.851	— 3.766	— 4.712
7	— 3.000	— 2.228	— 1.895	— 2.368	— 2.865	— 3.424
	1.000	1.000	1.000	1.000	1.000	1.000
	2.000	1.281	0.781	0.351	0.266	0.212

$N = 2 \quad \Omega = \frac{1}{2}$

base vectors: $|220+\rangle$, $|200+\rangle$, $|221-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
11	8.719	6.379	4.368	2.630	3.298	4.394
	1.000	1.000	1.000	1.000	1.000	1.000
	0.649	0.591	0.432	— 0.717	— 1.143	— 1.287
	— 0.428	— 0.605	— 0.907	— 1.066	— 0.675	— 0.454
9	2.568	1.693	0.667	0.120	— 0.237	— 0.853
	1.000	1.000	1.000	1.000	1.000	1.000
	2.203	2.227	2.828	— 15.696	15.901	6.635
	5.672	3.827	2.449	11.489	— 25.472	— 16.609
6	— 4.287	— 3.072	— 2.035	— 3.751	— 6.069	— 8.542
	1.000	1.000	1.000	1.000	1.000	1.000
	— 1.319	— 1.227	— 0.927	0.503	0.622	0.662
	0.336	0.453	0.662	0.600	0.428	0.325

$N = 3 \quad \Omega = \frac{7}{2}$

eigenvalue: $r = \eta - 7.2$, eigenvector: $|333+\rangle$.

	$\eta = -6$	-4	-2	2	4	6
10	— 13.200	— 11.200	— 9.200	— 5.200	— 3.200	— 1.200

$N = 3 \quad \Omega = \frac{5}{2}$

base vectors: $|332+\rangle$, $|333-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	— 4.200	— 3.200	— 1.828	1.572	3.424	5.321
15						
	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.817	— 1.225	— 1.785	— 3.173	— 3.929	— 4.703
	— 9.200	— 8.200	— 7.572	— 6.972	— 6.824	— 6.721
12						
	1.000	1.000	1.000	1.000	1.000	1.000
	1.225	0.817	0.560	0.315	0.255	0.213

$N = 3 \quad \Omega = \frac{3}{2}$

base vectors: $|331+\rangle$, $|311+\rangle$, $|332-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	3.058	1.614	0.483	0.381	2.091	3.967
19	1.000	1.000	1.000	1.000	1.000	1.000
	0.574	0.560	0.473	— 1.816	— 2.300	— 2.322
	— 0.601	— 0.829	— 1.178	— 1.225	— 0.737	— 0.513
	— 3.054	— 3.124	— 2.765	— 1.129	— 1.319	— 1.491
16	1.000	1.000	1.000	1.000	1.000	1.000
	2.137	3.280	11.837	2.543	1.585	1.416
	3.703	3.421	5.599	— 2.953	— 3.590	— 4.460
	— 9.104	— 7.590	— 6.819	— 8.352	— 9.872	— 11.576
13	1.000	1.000	1.000	1.000	1.000	1.000
	— 1.262	— 0.917	— 0.408	0.204	0.303	0.356
	0.458	0.587	0.685	0.514	0.412	0.337

$N = 3 \quad \Omega = \frac{1}{2}$

base vectors: $| 330 + \rangle$, $| 310 + \rangle$, $| 331 - \rangle$, $| 311 - \rangle$.

	$\eta = -6$	−4	−2	2	4	6
	10.824	7.100	3.594	2.512	4.265	6.126
26	1.000	1.000	1.000	1.000	1.000	1.000
	1.460	1.518	1.650	−1.495	−1.161	−1.049
	−0.474	−0.620	−0.854	−1.554	−2.053	−2.587
	−0.307	−0.449	−0.823	4.547	5.248	6.244
	3.620	1.779	0.257	−1.636	−2.151	−2.373
20	1.000	1.000	1.000	1.000	1.000	1.000
	−4.476	−3.201	−1.730	−0.645	−0.926	−0.950
	−8.469	−4.424	−1.803	−0.837	−0.468	−0.300
	−4.959	−2.482	−0.379	−0.718	−0.578	−0.444
	−2.465	−2.012	−1.275	−3.480	−5.829	−8.066
17	1.000	1.000	1.000	1.000	1.000	1.000
	−0.539	−0.379	0.328	7.020	5.302	5.955
	−0.030	−0.136	−0.197	−4.641	−6.453	−10.375
	0.740	1.134	2.076	0.503	−1.542	−3.458
	−7.779	−6.666	−6.376	−9.196	−12.086	−15.487
14	1.000	1.000	1.000	1.000	1.000	1.000
	−0.582	−0.484	−0.303	0.359	0.637	0.809
	1.432	1.088	0.918	0.777	0.632	0.500
	−1.718	−0.912	−0.347	0.164	0.198	0.183

$N = 4 \quad \Omega = \frac{9}{2}$

eigenvalue: $r = \frac{4}{3}\eta - 13$, eigenvector: $| 444 + \rangle$.

	$\eta = -6$	−4	−2	2	4	6
18	−21.000	−18.333	−15.667	−10.333	−7.667	−5.000

$N = 4 \quad \Omega = \frac{7}{2}$

base vectors: $| 443 + \rangle$, $| 444 - \rangle$.

	$\eta = -6$	−4	−2	2	4	6
	−10.628	−8.632	−6.392	−1.518	1.018	3.589
25	1.000	1.000	1.000	1.000	1.000	1.000
	−1.192	−1.662	−2.219	−3.470	−4.131	−4.804
	−16.372	−15.035	−13.942	−12.148	−11.351	−10.589
21	1.000	1.000	1.000	1.000	1.000	1.000
	0.839	0.602	0.451	0.288	0.242	0.208

$N = 4 \quad \Omega = \frac{5}{2}$

base vectors: $|442+\rangle$, $|422+\rangle$, $|443-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	— 3.373	— 3.871	— 4.087	— 2.217	0.277	2.838
31	1.000	1.000	1.000	1.000	1.000	1.000
	0.552	0.558	0.517	— 3.610	— 3.561	— 3.352
	— 0.809	— 1.081	— 1.451	— 1.201	— 0.757	— 0.547
	— 9.282	— 8.360	— 6.780	— 3.855	— 3.470	— 3.008
27	1.000	1.000	1.000	1.000	1.000	1.000
	2.465	5.272	90.857	1.119	0.946	0.914
	2.918	3.646	33.064	— 2.532	— 3.127	— 3.773
	— 15.045	— 13.469	— 12.833	— 13.628	— 14.506	— 15.530
22	1.000	1.000	1.000	1.000	1.000	1.000
	— 1.034	— 0.611	— 0.232	0.127	0.200	0.245
	0.531	0.610	0.607	0.451	0.380	0.325

$N = 4 \quad \Omega = \frac{3}{2}$

base vectors: $|441+\rangle$, $|421+\rangle$, $|442-\rangle$, $|422-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	4.911	2.230	0.171	2.276	4.651	7.148
42	1.000	1.000	1.000	1.000	1.000	1.000
	1.244	1.404	2.026	— 3.127	— 2.112	— 1.784
	— 0.629	— 0.809	— 1.071	— 1.816	— 2.246	— 2.689
	— 0.431	— 0.713	— 1.871	9.992	9.053	9.513
	— 2.239	— 3.332	— 3.800	— 4.457	— 4.200	— 3.765
33	1.000	1.000	1.000	1.000	1.000	1.000
	— 148.417	— 7.290	— 0.671	— 1.560	— 1.677	— 1.597
	— 208.425	— 7.611	— 1.393	— 0.961	— 0.566	— 0.379
	— 121.847	— 4.315	0.605	— 0.763	— 0.642	— 0.512
	— 7.119	— 5.521	— 4.335	— 5.846	— 7.524	— 9.133
29	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.545	— 0.233	3.603	2.001	1.780	1.835
	— 0.297	— 0.510	0.727	— 2.294	— 3.008	— 3.917
	1.181	1.524	4.020	0.109	— 0.441	— 0.868
	— 12.952	— 12.111	— 12.104	— 14.706	— 16.994	— 19.649
23	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.623	— 0.462	— 0.245	0.215	0.373	0.482
	0.969	0.829	0.763	0.627	0.537	0.456
	— 0.891	— 0.450	— 0.167	0.081	0.110	0.114

$N = 4 \quad \Omega = \frac{1}{2}$

base vectors: $|440+\rangle$, $|420+\rangle$, $|400+\rangle$, $|441-\rangle$, $|421-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	13.736	8.749	4.034	2.039	4.403	6.898
	1.000	1.000	1.000	1.000	1.000	1.000
51	2.266	2.555	3.410	— 4.291	— 3.337	— 2.987
	1.561	1.843	2.667	6.641	4.782	4.099
	— 0.511	— 0.639	— 0.835	— 0.814	— 0.613	— 0.488
	— 0.698	— 1.052	— 2.135	2.532	1.293	0.870
	5.508	2.566	0.188	— 1.353	— 1.477	— 1.202
	1.000	1.000	1.000	1.000	1.000	1.000
43	— 1.275	— 0.938	— 0.205	— 1.173	— 0.441	— 0.215
	— 2.191	— 2.308	— 3.435	— 2.734	— 1.886	— 1.691
	— 3.821	— 2.644	— 1.728	— 1.567	— 2.106	— 2.709
	— 4.813	— 3.764	— 3.474	4.285	4.063	4.563
	— 1.437	— 2.333	— 3.037	— 6.163	— 8.057	— 9.716
	1.000	1.000	1.000	1.000	1.000	1.000
34	0.085	0.172	0.136	— 0.504	— 0.644	— 0.589
	— 0.559	— 0.465	— 0.142	— 0.258	— 0.504	— 0.573
	— 0.290	— 0.472	— 0.841	— 0.816	— 0.438	— 0.261
	0.670	0.840	0.838	— 0.834	— 0.777	— 0.615
	— 6.733	— 5.465	— 4.165	— 7.293	— 11.045	— 14.793
	1.000	1.000	1.000	1.000	1.000	1.000
30	6.922	— 17.873	— 9.735	3.961	3.743	5.130
	— 9.733	20.639	7.376	1.714	2.139	3.283
	10.840	— 17.270	— 5.376	— 2.728	— 4.024	— 7.444
	— 5.800	4.197	— 3.760	0.947	— 0.933	— 3.180
	— 12.474	— 11.584	— 11.753	— 15.296	— 18.558	— 22.587
	1.000	1.000	1.000	1.000	1.000	1.000
24	— 0.868	— 0.554	— 0.263	0.267	0.531	0.758
	0.659	0.302	0.071	0.055	0.181	0.318
	0.791	0.892	0.914	0.853	0.770	0.658
	— 0.491	— 0.408	— 0.221	0.187	0.295	0.325

$N = 5 \quad \Omega = \frac{11}{2}$

eigenvalue: $\frac{5}{3}\eta - 18.5$, eigenvector: $|555+\rangle$.

	$\eta = -6$	-4	-2	2	4	6
28	— 28.500	— 25.177	— 21.833	— 15.177	— 11.833	— 8.500

$N = 5 \quad \Omega = \frac{9}{2}$

base vectors: $|554+\rangle$, $|555-\rangle$.

	η = —6	—4	—2	2	4	6
40	—16.500	—13.636	—10.616	—4.322	—1.105	2.139
	1.000	1.000	1.000	1.000	1.000	1.000
	—1.581	—2.065	—2.599	—3.745	—4.341	—4.946
32	—23.500	—21.698	—20.050	—17.011	—15.562	—14.139
	1.000	1.000	1.000	1.000	1.000	1.000
	0.632	0.484	0.385	0.267	0.230	0.202

$N = 5 \quad \Omega = \frac{7}{2}$

base vectors: $|553+\rangle$, $|533+\rangle$, $|554-\rangle$.

	η = —6	—4	—2	2	4	6
48	—9.480	—9.049	—8.384	—5.218	—2.054	1.176
	1.000	1.000	1.000	1.000	1.000	1.000
	0.497	0.490	0.433	—4.295	—4.425	—4.149
	—1.056	—1.361	—1.732	—1.439	—0.888	—0.636
41	—15.523	—13.722	—11.256	—6.609	—5.512	—4.366
	1.000	1.000	1.000	1.000	1.000	1.000
	2.442	5.528	37.895	1.145	0.870	0.814
	2.098	2.726	10.044	—2.723	—3.210	—3.741
35	—21.397	—19.628	—18.760	—18.573	—18.834	—19.210
	1.000	1.000	1.000	1.000	1.000	1.000
	—0.871	—0.461	—0.168	0.096	0.155	0.194
	0.537	0.569	0.535	0.408	0.354	0.309

$N = 5 \quad \Omega = \frac{5}{2}$

base vectors: $|552+\rangle$, $|532+\rangle$, $|553-\rangle$, $|533-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	— 1.116	— 2.676	— 3.119	1.371	4.423	7.580
61	1.000	1.000	1.000	1.000	1.000	1.000
	1.129	1.355	2.466	— 4.479	— 2.914	— 2.403
	— 0.816	— 1.030	— 1.306	— 1.992	— 2.368	— 2.753
	— 0.587	— 1.074	— 3.540	15.268	12.525	12.386
	— 8.258	— 8.569	— 7.442	— 7.416	— 6.489	— 5.388
50	1.000	1.000	1.000	1.000	1.000	1.000
	9.749	— 4.844	— 0.035	— 1.917	— 2.198	— 2.094
	10.595	— 4.599	— 1.386	— 1.184	— 0.692	— 0.463
	5.725	— 0.772	0.769	— 0.782	— 0.722	— 0.590
	— 11.586	— 9.284	— 8.749	— 8.607	— 9 501	— 10.393
44	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.454	0.350	21.498	1.944	1.461	1.413
	— 0.506	— 0.455	7.169	— 2.428	— 2.893	— 3.491
	1.534	1.809	12.613	0.188	— 0.287	— 0.582
	— 18.840	— 17.937	— 17.823	— 19.815	— 21.567	— 23.599
36	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.618	— 0.425	— 0.206	0.162	0.279	0.362
	0.785	0.714	0.667	0.546	0.479	0.419
	— 0.575	— 0.291	— 0.107	0.053	0.076	0.083

$N = 5 \quad \Omega = \frac{3}{2}$

base vectors: $|551+\rangle$, $|531+\rangle$, $|511+\rangle$, $|552-\rangle$, $|532-\rangle$.

	η = — 6	— 4	— 2	2	4	6
	7.808	3.907	0.568	0.918	3.947	7.097
	1.000	1.000	1.000	1.000	1.000	1.000
70	1.890	2.226	3.338	— 5.411	— 4.112	— 3.623
	0.976	1.232	2.039	12.795	8.865	7.343
	— 0.667	— 0.827	— 1.048	— 0.987	— 0.740	— 0.586
	— 0.870	— 1.390	— 3.159	4.393	2.149	1.407
	— 0.191	— 2.274	— 3.276	— 1.865	— 1.323	— 0.438
	1.000	1.000	1.000	1.000	1.000	1.000
62	— 4.359	— 2.719	— 1.115	— 2.461	— 1.306	— 0.946
	— 4.856	— 4.512	— 8.014	— 3.718	— 2.318	— 1.948
	— 7.139	— 3.871	— 2.218	— 1.717	— 2.134	— 2.583
	— 8.290	— 5.329	— 5.299	7.183	5.864	5.942
	— 6.357	— 6.436	— 6.565	— 9.164	— 10.375	— 11.357
	1.000	1.000	1.000	1.000	1.000	1.000
52	— 0.048	0.071	0.052	— 0.823	— 1.124	— 1.077
	— 0.500	— 0.339	— 0.046	— 0.235	— 0.483	— 0.566
	— 0.467	— 0.708	— 1.099	— 1.036	— 0.578	— 0.351
	0.841	0.953	0.707	— 0.790	— 0.822	— 0.678
	— 11.233	— 9.299	— 7.877	— 10.230	— 12.980	— 15.877
	1.000	1.000	1.000	1.000	1.000	1.000
46	2.985	13.310	141.387	3.477	2.431	2.483
	— 5.559	— 17.369	— 85.487	0.887	0.858	0.997
	4.330	11.111	59.345	— 2.665	— 3.114	— 4.047
	— 1.922	0.032	74.877	0.874	— 0.423	— 1.205
	— 17.727	— 16.931	—17. 216	— 20.692	— 23.635	— 27.126
	1.000	1.000	1.000	1.000	1.000	1.000
37	— 0.754	— 0.475	— 0.229	0.214	0.399	0.546
	0.546	0.211	0.043	0.027	0.081	0.138
	0.780	0.810	0.794	0.701	0.631	0.555
	— 0.474	— 0.337	— 0.161	0.116	0.180	0.205

$N = 5 \quad \Omega = \frac{1}{2}$

base vectors: $|550+\rangle$, $|530+\rangle$, $|510+\rangle$, $|551-\rangle$, $|531-\rangle$, $|511-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	17.207	10.952	5.017	3.162	6.118	9.237
	1.000	1.000	1.000	1.000	1.000	1.000
	2.944	3.426	4.884	−4.742	−3.082	−2.542
A	3.717	4.688	8.018	6.313	3.072	2.252
	−0.558	−0.681	−0.857	−1.391	−1.720	−2.061
	−1.148	−1.723	−3.469	8.472	7.414	7.644
	−0.632	−1.081	−3.043	−23.071	−16.614	−15.753
	8.305	4.175	0.577	−2.149	−1.654	−0.781
	1.000	1.000	1.000	1.000	1.000	1.000
71	−0.069	0.327	1.517	−3.818	−2.730	−2.359
	−2.803	−3.133	−4.467	3.716	2.449	1.981
	−2.853	−2.132	−1.506	−0.822	−0.606	−0.469
	−5.438	−4.780	−4.972	2.074	0.780	0.420
	−2.826	−2.665	−2.914	2.656	1.431	0.993
	0.663	−1.195	−1.835	−4.868	−6.961	−8.308
	1.000	1.000	1.000	1.000	1.000	1.000
63	0.743	1.011	2.203	−1.007	0.082	0.438
	−0.683	−0.652	−0.455	−4.445	−2.549	−2.161
	−0.402	−0.559	−0.768	−1.545	−2.077	−2.735
	0.384	0.310	−0.524	4.172	3.179	3.313
	0.681	1.157	3.480	0.659	1.207	1.649
	−5.968	−6.278	−6.309	−10.324	−13.537	−16.571
	1.000	1.000	1.000	1.000	1.000	1.000
53	−5.341	−2.350	−0.424	−0.092	−0.353	−0.303
	3.586	1.279	0.121	−0.085	−0.464	−0.622
	−7.107	−2.545	−1.091	−0.946	−0.499	−0.262
	0.251	−0.040	0.401	−0.678	−0.887	−0.735
	3.619	0.691	−0.184	−0.153	−0.305	−0.299
	−9.867	−7.850	−7.135	−11.269	−15.732	−20.742
	1.000	1.000	1.000	1.000	1.000	1.000
47	−0.482	0.068	4.020	15.520	5.093	6.911
	0.138	−0.175	−1.629	7.570	4.087	6.789
	−0.073	−0.161	0.987	−7.545	−4.822	−8.888
	0.846	1.192	3.905	8.387	−0.393	−3.757
	−1.326	−1.419	−2.239	2.459	0.195	−0.742
	−16.940	−16.403	−16.915	−21.152	−24.834	−29.436
	1.000	1.000	1.000	1.000	1.000	1.000
38	−0.654	−0.458	−0.234	0.237	0.479	0.712
	0.260	0.140	0.041	0.046	0.183	0.379
	0.994	0.944	0.928	0.888	0.843	0.765
	−0.724	−0.440	−0.211	0.192	0.343	0.428
	0.502	0.188	0.039	0.024	0.071	0.110

$N = 6 \quad \Omega = \frac{13}{2}$

eigenvalue: $2\eta - 24.9$, eigenvector: $|666+\rangle$.

	$\eta = -6$	−4	−2	2	4	6
39	−36.900	−32.900	−28.900	−20.900	−16.900	−12.900

$N = 6 \quad \Omega = \frac{11}{2}$

base vectors: $|665+\rangle$, $|666-\rangle$.

	$\eta = -6$	−4	−2	2	4	6
56	−23.128 1.000 −1.955	−19.476 1.000 −2.432	−15.721 1.000 −2.938	−8.035 1.000 −4.002	−4.139 1.000 −4.550	−0.221 1.000 −5.103
45	−31.672 1.000 0.512	−29.324 1.000 0.411	−27.079 1.000 0.340	−22.766 1.000 0.250	−20.661 1.000 0.220	−18.579 1.000 0.196

$N = 6 \quad \Omega = \frac{9}{2}$

base vectors: $|664+\rangle$, $|644+\rangle$, $|665-\rangle$.

	$\eta = -6$	−4	−2	2	4	6
66	−16.305 1.000 0.454 −1.305	−15.010 1.000 0.437 −1.623	−13.535 1.000 0.371 −1.983	−9.123 1.000 −4.776 −1.689	−5.290 1.000 −5.265 −1.017	−1.392 1.000 −4.938 −0.721
59	−22.640 1.000 2.719 1.712	−19.912 1.000 6.371 2.332	−16.602 1.000 33.063 6.685	−10.291 1.000 1.240 −2.915	−8.490 1.000 0.833 −3.325	−6.661 1.000 0.755 −3.785
49	−28.854 1.000 −0.698 0.524	−26.878 1.000 −0.348 0.522	−25.663 1.000 −0.127 0.480	−24.386 1.000 0.077 0.376	−24.020 1.000 0.126 0.332	−23.747 1.000 0.159 0.296

$N = 6 \quad \Omega = \frac{7}{2}$

base vectors: $|663+\rangle$, $|643+\rangle$, $|664-\rangle$, $|644-\rangle$.

	η = — 6	— 4	— 2	2	4	6
	— 7.780	— 8.110	— 6.895	— 0.473	3.260	7.085
	1.000	1.000	1.000	1.000	1.000	1.000
	1.099	1.453	3.248	— 5.916	— 3.740	— 3.025
	— 1.007	— 1.240	— 1.517	— 2.154	— 2.493	— 2.838
	— 0.810	— 1.672	— 6.277	21.577	16.514	15.604
	— 15.209	— 13.845	— 12.205	— 11.281	— 9.688	— 7.920
67	1.000	1.000	1.000	1.000	1.000	1.000
	9.127	— 0.484	0.120	— 2.138	— 2.684	— 2.570
	8.182	— 1.454	— 1.565	— 1.409	— 0.815	— 0.543
	3.443	1.255	0.600	— 0.773	— 0.791	— 0.661
	— 16.823	— 14.934	— 14.132	— 12.307	— 12.452	— 12.639
64	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.203	4.223	25.832	2.081	1.317	1.216
	— 0.627	1.807	7.150	— 2.594	— 2.914	— 3.375
	1.739	2.926	11.799	0.265	— 0.202	— 0.442
	— 25.988	— 24.910	— 24.569	— 25.738	— 26.920	— 28.326
54	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.574	— 0.372	— 0.171	0.129	0.222	0.290
	0.684	0.639	0.598	0.493	0.440	0.392
	— 0.395	— 0.200	— 0.073	0.038	0.056	0.063

$N = 6 \quad \Omega = \frac{5}{2}$

base vectors: $|662+\rangle$, $|642+\rangle$, $|622+\rangle$, $|663-\rangle$, $|643-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	1.230	— 1.526	— 3.467	— 1.132	2.568	6.381
	1.000	1.000	1.000	1.000	1.000	1.000
	1.758	2.180	3.683	— 6.606	— 4.943	— 4.302
	0.796	1.074	1.971	20.348	13.926	11.254
	— 0.826	— 1.008	— 1.239	— 1.151	— 0.856	— 0.676
	— 1.104	— 1.867	— 4.645	6.763	3.192	2.042
	— 6.774	— 7.855	— 7.295	— 3.487	— 2.228	— 0.688
	1.000	1.000	1.000	1.000	1.000	1.000
	— 16.275	— 5.331	— 2.222	— 3.633	— 2.035	— 1.525
	— 16.446	— 9.117	— 18.549	— 4.837	— 2.748	— 2.197
	— 19.525	— 5.516	— 2.787	— 1.860	— 2.210	— 2.583
	— 22.252	— 7.957	— 8.672	10.540	7.928	7.548
	— 12.017	— 11.415	— 11.176	— 13.048	— 13.595	— 13.901
	1.000	1.000	1.000	1.000	1.000	1.000
72	— 0.078	0.047	0.058	— 1.004	— 1.509	— 1.477
	— 0.408	— 0.204	— 0.035	— 0.196	— 0.452	— 0.546
	— 0.658	— 0.940	— 1.296	— 1.240	— 0.706	— 0.434
	0.980	0.980	0.592	— 0.750	— 0.867	— 0.739
	— 16.668	— 14.185	— 12.795	— 14.039	— 15.961	— 18.090
	1.000	1.000	1.000	1.000	1.000	1.000
65	2.503	7.250	26.193	3.590	2.013	1.874
	— 4.837	— 8.414	— 10.944	0.644	0.519	0.560
	2.984	4.856	8.652	— 2.776	— 2.920	— 3.460
	— 0.829	1.543	14.032	0.949	— 0.244	— 0.775
	— 24.271	— 23.519	— 23.767	— 26.794	— 29.285	— 32.203
	1.000	1.000	1.000	1.000	1.000	1.000
55	— 0.658	— 0.414	— 0.197	0.173	0.316	0.427
	0.393	0.138	0.027	0.015	0.046	0.078
	0.739	0.736	0.708	0.615	0.558	0.500
	— 0.413	— 0.266	— 0.119	0.080	0.126	0.148

$N = 6 \quad \Omega = \frac{3}{2}$

base vectors: $|661+\rangle$, $|641+\rangle$, $|621+\rangle$, $|662-\rangle$, $|642-\rangle$, $|622-\rangle$.

	η = — 6	— 4	— 2	2	4	6
	10.761	5.658	1.285	3.270	6.868	10.631
	1.000	1.000	1.000	1.000	1.000	1.000
	2.571	3.121	4.948	— 6.377	— 4.023	— 3.253
	2.483	3.385	7.184	14.197	6.316	4.398
	— 0.698	— 0.842	— 1.033	— 1.533	— 1.819	— 2.115
	— 1.376	— 2.161	— 4.760	12.321	9.797	9.519
	— 0.698	— 1.344	— 5.119	— 49.751	— 29.946	— 25.832
	2.006	— 1.194	— 3.531	— 4.003	— 2.832	— 1.320
	1.000	1.000	1.000	1.000	1.000	1.000
	— 1.497	— 0.651	1.655	— 4.837	— 3.465	— 2.970
	— 5.024	— 5.355	— 5.770	6.958	4.680	3.719
	— 4.106	— 2.721	— 1.564	— 0.987	— 0.724	— 0.560
	— 7.338	— 5.968	— 5.286	3.693	1.462	0.817
	— 3.381	— 2.952	— 1.072	3.571	2.008	1.393
	— 4.789	— 5.332	— 4.347	— 6.198	— 7.446	— 8.179
	1.000	1.000	1.000	1.000	1.000	1.000
	0.607	1.054	3.209	— 2.192	— 0.817	— 0.376
	— 0.714	— 0.674	1.207	— 5.616	— 3.098	— 2.487
	— 0.553	— 0.728	— 0.825	— 1.679	— 2.075	— 2.532
	0.396	0.124	— 1.140	6.616	4.690	4.435
	0.899	1.749	6.218	0.389	1.150	1.504
	— 11.500	— 10.961	— 10.573	— 14.358	— 16.884	— 19.214
	1.000	1.000	1.000	1.000	1.000	1.000
73	— 10.085	— 0.974	— 0.132	— 0.345	— 0.737	— 0.738
	8.500	0.511	0.012	— 0.138	— 0.485	— 0.652
	— 12.091	— 1.431	— 1.153	— 1.114	— 0.646	— 0.360
	1.047	0.682	0.508	— 0.659	— 0.893	— 0.779
	4.543	— 0.432	— 0.155	— 0.104	— 0.223	— 0.237
	— 13.717	— 11.779	— 11.794	— 15.386	— 18.894	— 22.914
	1.000	1.000	1.000	1.000	1.000	1.000
68	— 0.279	1.453	11.255	8.624	3.440	3.212
	— 0.030	— 1.143	— 4.328	2.757	1.747	1.977
	— 0.185	0.605	3.513	— 4.475	— 3.609	— 4.429
	1.012	1.625	8.045	3.864	— 0.089	— 1.321
	— 1.508	— 1.752	— 3.188	0.796	0.130	— 0.153
	— 22.962	— 22.593	— 23.240	— 27.524	— 31.011	— 35.204
	1.000	1.000	1.000	1.000	1.000	1.000
57	— 0.628	— 0.424	— 0.211	0.205	0.395	0.562
	0.273	0.131	0.033	0.029	0.103	0.196
	0.865	0.840	0.819	0.751	0.699	0.636
	— 0.546	— 0.340	— 0.160	0.132	0.225	0.278
	0.302	0.108	0.021	0.012	0.033	0.051

$N = 6 \quad \Omega = \frac{1}{2}$

base vectors: $|\,660+\rangle$, $|\,640+\rangle$, $|\,620+\rangle$, $|\,600+\rangle$, $|\,661-\rangle$, $|\,641-\rangle$, $|\,621-\rangle$.

	$\eta = -6$	— 4	— 2	2	4	6
	20.703	13.179	6.009	3.047	6.636	10.396
	1.000	1.000	1.000	1.000	1.000	1.000
	3.642	4.348	6.495	— 7.352	— 5.077	— 4.300
	6.358	8.496	16.462	23.764	12.993	9.905
	4.037	5.649	12.004	— 34.171	— 17.158	— 12.525
	— 0.598	— 0.714	— 0.874	— 0.862	— 0.697	— 0.581
	— 1.658	— 2.483	— 4.971	5.484	2.800	1.888
	— 1.709	— 3.009	— 8.870	— 12.174	— 4.392	— 2.551
	11.182	5.874	1.264	— 1.158	— 0.286	1.113
	1.000	1.000	1.000	1.000	1.000	1.000
	0.910	1.476	3.282	— 4.449	— 2.472	— 1.857
	— 2.356	— 2.508	— 2.531	2.535	0.142	— 0.273
	— 2.769	— 3.742	— 8.776	9.588	4.365	3.219
	— 2.380	— 1.866	— 1.416	— 1.365	— 1.696	— 2.056
	— 6.154	— 5.848	— 6.970	7.890	6.323	6.331
	— 5.976	— 6.372	— 10.012	— 15.544	— 9.244	— 8.186
	2.886	— 0.154	— 2.437	— 6.473	— 7.838	— 8.605
	1.000	1.000	1.000	1.000	1.000	1.000
	1.430	1.862	3.257	— 3.672	— 2.368	— 1.936
	— 0.190	0.017	0.545	1.983	1.119	0.736
	— 0.864	— 0.983	— 0.980	1.343	1.251	1.124
	— 0.482	— 0.629	— 0.855	— 0.853	— 0.613	— 0.462
	— 0.015	— 0.345	— 1.901	2.129	0.521	0.147
	1.068	1.660	3.333	3.422	1.818	1.208
	— 4.471	— 5.273	— 4.443	— 8.425	— 12.460	— 15.518
	1.000	1.000	1.000	1.000	1.000	1.000
	— 2.275	— 1.386	0.225	— 1.078	0.351	0.879
B	— 1.051	— 2.133	— 7.489	— 6.249	— 2.765	— 2.021
	3.089	3.545	7.386	— 3.250	— 1.945	— 1.712
	— 3.756	— 2.417	— 1.749	— 1.581	— 2.058	— 2.746
	— 2.325	— 2.538	— 5.201	4.927	2.906	2.621
	2.694	1.631	— 0.539	— 0.011	1.419	2.034
	— 10.435	— 10.131	— 10.266	— 15.117	— 19.353	— 23.659
	1.000	1.000	1.000	1.000	1.000	1.000
	— 0.163	— 0.148	— 0.235	0.113	— 0.113	— 0.121
74	— 0.235	— 0.036	0.060	0.009	— 0.256	— 0.482
	0.296	0.032	— 0.026	0.003	— 0.116	— 0.268
	— 0.322	— 0.685	— 1.033	— 1.021	— 0.626	— 0.299
	0.853	0.885	0.418	— 0.493	— 0.892	— 0.816
	— 0.653	— 0.493	— 0.115	— 0.125	— 0.416	— 0.495

Continued next page

Continuation of

$N = 6 \quad \Omega = \frac{1}{2}$

base vectors: $|\,660+\rangle$, $|\,640+\rangle$, $|\,620+\rangle$, $|\,600+\rangle$, $|\,661-\rangle$, $|\,641-\rangle$, $|\,621-\rangle$.

	η = — 6	— 4	— 2	2	4	6
	— 13.673	— 11.555	— 11.349	— 16.175	— 20.943	— 26.907
	1.000	1.000	1.000	1.000	1.000	1.000
69	9.085	10.291	6.560	— 14.612	12.536	10.379
	— 13.400	— 9.981	— 2.819	— 6.223	10.248	11.336
	12.883	7.570	1.088	— 1.686	4.288	5.538
	11.185	7.818	1.830	4.984	— 10.418	— 12.297
	— 3.429	3.850	5.694	— 10.813	2.228	— 4.031
	— 0.061	— 3.798	— 2.214	— 3.733	2.374	— 0.462
	— 22.392	— 22.141	— 22.979	— 27.899	— 31.957	— 37.020
	1.000	1.000	1.000	1.000	1.000	1.000
60	— 0.658	— 0.432	— 0.216	0.217	0.436	0.657
	0.356	0.149	0.036	0.037	0.151	0.337
	— 0.209	— 0.059	— 0.007	0.006	0.041	0.120
	0.934	0.943	0.937	0.910	0.883	0.836
	— 0.582	— 0.397	— 0.197	0.187	0.354	0.480
	0.251	0.121	0.031	0.027	0.093	0.170

Table I a. Eigenvalues for the Spherical Case ($\delta = 0$).

Level designation in the spherical case		r	Label on level in Fig. 5
$N = 0$	$s_{1/2}$	0.000	
$N = 1$	$p_{3/2}$	— 1.000	2, 3
	$p_{1/2}$	2.000	4
$N = 2$	$s_{1/2}$	0.000	9
	$d_{5/2}$	— 2.000	5, 6, 7
	$d_{3/2}$	3.000	8, 11
$N = 3$	$p_{3/2}$	— 1.700	16, 17
	$p_{1/2}$	1.300	26
	$f_{7/2}$	— 7.200	10, 12, 13, 14
	$f_{5/2}$	— 0.200	15, 19, 20
$N = 4$	$s_{1/2}$	0.000	43
	$d_{5/2}$	— 4.700	27, 29, 30
	$d_{3/2}$	0.300	42, 51
	$g_{9/2}$	— 13.000	18, 21, 22, 23, 24
	$g_{7/2}$	— 4.000	25, 31, 33, 34
$N = 5$	$p_{3/2}$	— 1.900	70, 71
	$p_{1/2}$	1.100	A
	$f_{7/2}$	— 8.400	41, 44, 46, 47
	$f_{5/2}$	— 1.400	61, 62, 63
	$h_{11/2}$	— 18.500	28, 32, 35, 36, 37, 38
	$h_{9/2}$	— 7.500	40, 48, 50, 52, 53
$N = 6$	$s_{1/2}$	0.000	
	$d_{5/2}$	— 4.700	B
	$d_{3/2}$	0.300	
	$g_{9/2}$	— 13.000	59, 64, 65, 58, 69
	$g_{7/2}$	— 4.000	
	$i_{13/2}$	— 24.900	39, 45, 49, 54, 55, 57, 60
	$i_{11/2}$	— 11.900	56, 66, 67, 72, 73, 74
$N = 7$	— — —		
	$j_{15/2}$	— 29.400	58
	— — —		

TABLE Ib. Eigenvalues and Eigenfunctions of the Shell $N = 4$ with the Parameter $\mu = 0.55$ (added in proof).

$\Omega = \frac{9}{2}$

eigenvalue: $r = \frac{4}{3}\eta - 15$, eigenvector: $|444+\rangle$.

	$\eta = -6$	-4	-2	2	4	6
18	−23.000	−20.333	−17.667	−12.333	−9.667	−7.000

$\Omega = \frac{7}{2}$

base vectors: $|443+\rangle$, $|444-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
25	−12.628	−10.632	−8.392	−3.519	−0.982	1.589
	−0.643	−0.516	−0.411	−0.277	−0.235	−0.204
	0.766	0.857	0.912	0.961	0.972	0.979
21	−18.372	−17.035	−15.942	−14.148	−13.351	−12.589
	0.766	0.857	0.912	0.961	0.972	0.979
	0.643	0.516	0.411	0.277	0.235	0.204

$\Omega = \frac{5}{2}$

base vectors: $|442+\rangle$, $|422+\rangle$, $|443-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
31	−5.119	−5.644	−5.891	−2.948	−0.447	2.111
	0.705	−0.625	−0.521	−0.179	−0.228	−0.257
	0.458	−0.455	−0.485	0.972	0.964	0.959
	−0.541	0.634	0.702	0.152	0.137	0.119
27	−10.702	−9.435	−7.619	−5.741	−5.390	−4.948
	0.139	0.016	−0.138	−0.383	−0.316	−0.266
	0.659	0.805	0.860	−0.211	−0.207	−0.188
	0.739	0.593	0.492	0.899	0.926	0.946
22	−16.479	−15.221	−14.790	−15.611	−16.463	−17.463
	−0.695	0.780	0.842	0.906	0.921	0.929
	0.596	−0.381	−0.159	0.102	0.167	0.211
	−0.401	0.496	0.515	0.410	0.352	0.302

$\Omega = \frac{3}{2}$

base vectors: $|441+\rangle$, $|421+\rangle$, $|442-\rangle$, $|422-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	3.732	1.135	— 0.667	1.631	3.969	6.444
42	0.531	0.443	0.252	0.078	0.091	0.089
	0.746	0.745	0.684	— 0.292	— 0.218	— 0.175
	— 0.316	— 0.338	— 0.259	— 0.145	— 0.210	— 0.246
	— 0.249	— 0.368	— 0.634	0.942	0.949	0.949
	— 3.575	— 4.513	— 4.994	— 5.389	— 5.138	— 4.732
	— 0.031	— 0.063	0.154	— 0.311	— 0.409	— 0.464
33	0.490	0.579	0.660	0.870	0.851	0.841
	0.712	0.604	0.112	0.198	0.180	0.148
	0.501	0.544	0.727	0.326	0.275	0.237
	— 8.286	— 6.688	— 5.564	— 7.513	— 9.237	— 10.881
	0.595	0.544	— 0.553	— 0.448	— 0.344	— 0.269
29	— 0.300	— 0.165	0.266	— 0.364	— 0.386	— 0.355
	— 0.259	— 0.421	0.752	0.815	0.846	0.876
	0.699	0.707	— 0.240	0.050	0.132	0.187
	— 14.471	— 13.868	— 14.041	— 16.661	— 18.861	— 21.431
	— 0.602	0.710	0.779	0.834	0.840	0.840
23	0.336	— 0.286	— 0.163	0.157	0.280	0.369
	— 0.571	0.587	0.596	0.525	0.456	0.388
	0.446	— 0.264	— 0.109	0.060	0.084	0.090

$\Omega = \frac{1}{2}$

base vectors: $|\,440+\rangle$, $|\,420+\rangle$, $|\,400+\rangle$, $|\,441-\rangle$, $|\,421-\rangle$.

	$\eta = -6$	-4	-2	2	4	6
	13.130	8.188	3.551	1.811	4.142	6.614
	0.298	0.248	0.159	0.091	0.140	0.170
51	0.745	0.725	0.670	— 0.478	— 0.531	— 0.556
	0.537	0.559	0.596	0.833	0.810	0.796
	— 0.144	— 0.149	— 0.125	— 0.068	— 0.079	— 0.077
	— 0.218	— 0.282	— 0.393	0.253	0.189	0.151
	4.517	1.723	— 0.319	— 1.955	— 2.266	— 2.091
	— 0.142	— 0.166	— 0.146	0.150	0.176	0.163
43	0.152	0.116	— 0.064	— 0.271	— 0.123	— 0.062
	0.318	0.425	0.635	— 0.437	— 0.343	— 0.279
	0.531	0.443	0.256	— 0.235	— 0.373	— 0.445
	0.757	0.763	0.711	0.811	0.835	0.833
	— 2.734	— 3.610	— 4.315	— 7.297	— 9.189	— 10.887
	0.721	— 0.651	0.438	— 0.412	— 0.578	— 0.653
34	0.132	— 0.242	0.539	0.669	0.539	0.477
	— 0.458	0.423	— 0.394	0.289	0.370	0.414
	— 0.219	0.286	— 0.207	0.192	0.204	0.158
	0.454	— 0.507	0.565	0.511	0.444	0.387
	— 7.698	— 6.293	— 5.173	— 8.602	— 12.320	— 16.125
	— 0.050	0.194	— 0.492	— 0.498	— 0.305	— 0.187
30	0.450	— 0.552	0.479	— 0.470	— 0.549	— 0.493
	— 0.553	0.553	— 0.292	— 0.172	— 0.281	— 0.289
	0.628	— 0.574	0.664	0.707	0.702	0.741
	— 0.308	0.151	— 0.036	0.047	0.181	0.299
	— 13.815	— 13.275	— 13.676	— 17.224	— 20.301	— 24.111
	0.608	0.671	0.720	0.743	0.723	0.695
24	— 0.450	— 0.314	— 0.162	0.169	0.332	0.466
	0.308	0.149	0.037	0.031	0.103	0.183
	0.505	0.608	0.660	0.635	0.565	0.471
	— 0.280	— 0.243	— 0.137	0.121	0.192	0.211

TABLE II. Matrix Elements of the Coupling Energy between some Particular States with N Differing by Two. (The energy unit is $\varkappa \hbar \omega_0$.)

	$\eta = 2$	$\eta = 4$	$\eta = 6$
$\frac{1}{\varkappa \hbar \omega_0} \cdot \langle N = 4\ \Omega = \frac{3}{2}, \#42 \mid H_\delta \mid N = 6\ \Omega = \frac{3}{2}, \#57\rangle$	0.006	0.015	0.017
$\frac{1}{\varkappa \hbar \omega_0} \cdot \langle N = 4\ \Omega = \frac{1}{2}, \#51 \mid H_\delta \mid N = 6\ \Omega = \frac{1}{2}, \#60\rangle$	0.007	0.013	0.018

TABLE III. Connection between Ground State Spin I_0 and Decoupling Factor a.

Range of a	I_0
— — — —	— — — —
$-14 < a < -10$	11/2
$-10 < a < -6$	7/2
$-6 < a < -1$	3/2
$-1 < a < 4$	1/2
$4 < a < 8$	5/2
$8 < a < 12$	9/2
$12 < a < 16$	13/2
— — — —	— — — —

Indleveret til selskabet den 10. december 1954.
Færdig fra trykkeriet den 25. oktober 1955.

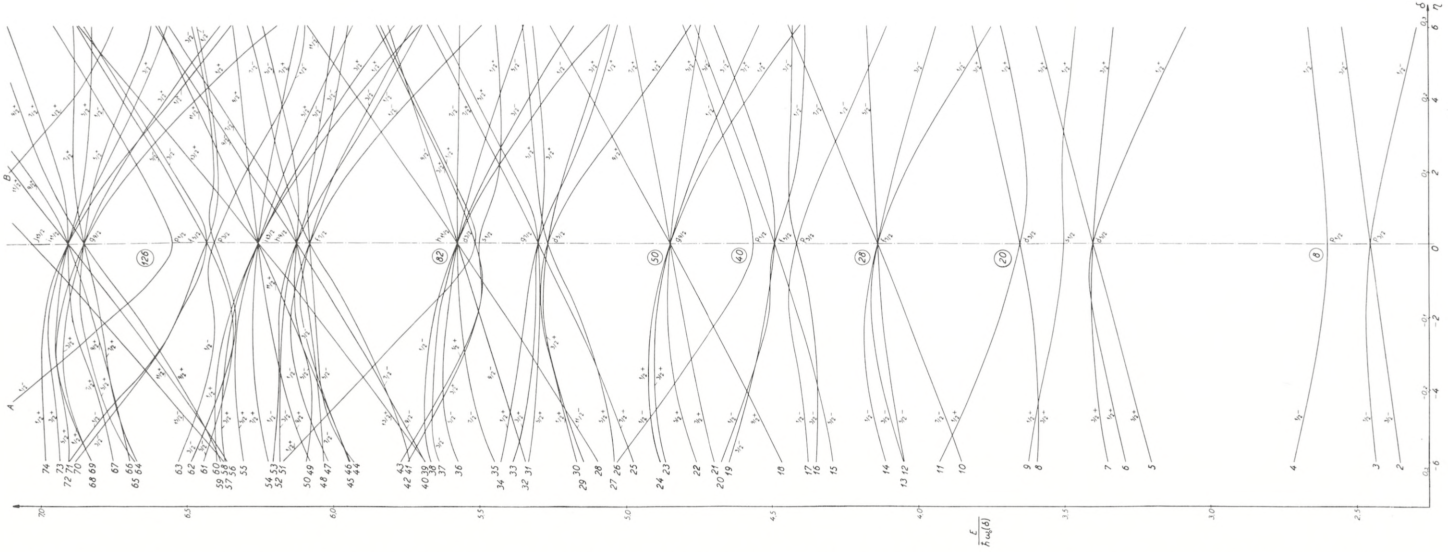

Fig. 5. *Energy levels of the model as functions of the deformation.* There is a scale for both of the deformation variables η and δ. To first order δ equals the deformation parameter β used in the papers of A. Bohr and B. Mottelson. The energy is plotted in the A- and δ-dependent energy unit $\varkappa\hbar\omega_0(\delta)$, where $\hbar\omega_0$ varies with A as $A^{-1/3}$. Further, for $A \approx 100$ one expects $\hbar\omega_0 \approx 8.8$ MeV. The constant $\varkappa$ is chosen equal to 0.05 in the diagram. The levels are labelled by the Ω-number and the parity. The numbers to the left of the curves refer to Table I.